A TREATISE

ON

PLANE TRIGONOMETRY.

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ON

PLANE TRIGONOMETRY,

CONTAINING

AN ACCOUNT OF HYPERBOLIC FUNCTIONS,

WITH

Numerous Examples.

BY

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PREFACE.

In the present Treatise my object has been to produce a work on Trigonometry fit to take its place with recent advanced text-books on other branches of Mathematics—a work in which the demonstrations will be accurate and complete, and the various parts of the subject, while avoiding too much detail, fully and comprehensively treated and properly co-ordinated.

A cursory examination will show that it contains not only everything that is usually given in books on Trigonometry, but also much that has hitherto appeared only in Mathematical periodicals.

I have given a systematic account of Imaginary Angles and Hyperbolic Functions; the latter, from their great analogy to Circular Functions, are very interesting, and their great and increasing importance, not only in Pure Mathematics but in Mathematical Physics, makes it essential that the student should become acquainted with them.

In compiling the work, the writers to whom I am principally indebted are Serret and Laisant, and, to a less extent, Mansion, Briot and Bouquet, and Hymers.

The Exercises are very numerous and carefully selected. Besides the usual sources (the Examination Papers set at the various Universities), some are from GLAISHER's Papers, and from WOLSTENHOLME's Mathematical Problems. Those on the Area of a Triangle (pp. 145 and 150) are from MATHESIS and from the *Annals of Mathematics*, edited by Professors STONE and THORNTON, of the University of Virginia, U. S. America. Several are original, and for a large number I am indebted to my mathematical friends, M. JOSEPH NEUBERG, Professor Ordinarius of the University of Liége, and Mr. M'CAY, F.T.C.D.

In conclusion, I have to return my best thanks to Professor NEUBERG, for many valuable suggestions and annotations received from him on a subject of which he is a master.

JOHN CASEY.

86, SOUTH CIRCULAR ROAD, DUBLIN.
March 25, 1888.

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ERRATA.

Page	2,	line	23,	*for* quantity *read* number.
,,	6,	,,	21,	*for* $75''$ *read* $0''\cdot75$.
,,	17,	,,	3,	*dele parenthesis after* $\sin\theta$.
,,	18,	last line,		*for* 0 *read* θ.
,,	22,		Ex. 23,	*put comma after* $\dfrac{1}{\sqrt{2}}$.
,,	23,	line	6,	*for* multiple of arcs *read* multiple arcs.
,,	23,	,,	7,	*for* submultiple of arcs *read* submultiple arcs.
,,	25,	,,	13,	*for* $1 + 2\sin\theta\cos\theta$ *read* $1 - 2\sin\theta\cos\theta$.
,,	28,	,,	18,	*for* $\sec 45^\circ + \tan 45^\circ)^2$ *read* $(\sec 45^\circ + \tan 45^\circ)^2$.
,,	39,	,,	1,	*for* $(\alpha + \beta + \gamma$ *read* $(\alpha + \beta + \gamma)$.
,,	39,	,,	23,	*for* (roman s_3) *read* (s_3 ital.).
,,	45,	,,	2,	*for* $\sin x(1 + \tan x/2\tan x$ *read* $\sin x(1 + \tan x)/2\tan x$.
,,	46,	,,	14,	*put a comma between* $\dfrac{x}{3}$ *and* $\sin\dfrac{x}{3^2}$.
,,	50,	,,	12,	*for* $(2n\pi + 1)\pi$ *read* $(2n + 1)\pi$.
,,	55,	,,	7,	*for* $+ 6\sin\left(2\theta + \dfrac{\pi}{3}\right)$ *read* $- 6\sin\left(2\theta + \dfrac{\pi}{3}\right)$.
,,	62,	,,	3,	*for* π *read* $\pi/4$.
,,	64,	,,	1,	*for* $\pi/3$, *read* $\pi/13$.
,,	70,	,,	9,	*for* $= \tan^2\phi - 3\tan^4\phi$ *read* $= - \tan^2\phi - 3\tan^4\phi$.
,,	71,	,,	14,	*for* $\dfrac{by}{\cos\theta}$ *read* $\dfrac{by}{\sin\theta}$.
,,	71,	,,	20,	*for* $a\sin\theta$ *read* $a\sin\phi$.
,,	92,	,,	17,	*after* infinity *put* is equal to zero.
,,	103,	,,	11,	*for* arcs from 45° to 90° *read* arcs between 45° and 90° to arcs between 0° and 45°.
,,	106,	,,	7,	*for* constants *read* constants of relation.
,,	110,	,,	17,	*for* deduce *read* deduce them.
,,	117,	,,	14,	*for* § 89, note, *read* § 89*a*.
,,	126,	,,	12,	*for* ,, *read* prove.
,,	127,	,,	5,	*for* $-b$ *read* $=b$.
,,	160,	,,	20,	*for* $3 + \sqrt{5}$ *read* $3(1 + \sqrt{5})$.
,,	161,	,,	26,	*for* the least angle *read* half the least angle.
,,	191,	,,	3,	*for* $\ddot{=}$ *read* $-$.
,,	191,	,,	16,	*for* IE' *read* IE^2.
,,	208,	,,	4,	*for* $e^{2\theta}$ *read* $e^{i\theta}$.
,,	208,	,,	9,	*put comma after* $r\cos\theta$.
,,	213,	,,	11,	*for* $e\theta$ *read* e^θ.
,,	238,	,,	9,	*for* $\operatorname{cosec}\alpha$ *read* $\sec\alpha$.
,,	238,	,,	2,	*for* $3\sec^2 4\alpha$ *read* $4\sec^2 4\alpha$.

A TREATISE ON TRIGONOMETRY.

CHAPTER I.

FIRST NOTIONS ON TRIGONOMETRIC FUNCTIONS.

SECTION I.—ON ARCS.

1. **The Numerical Measure** *of any quantity, such as an angle, a line, &c., is the ratio which it bears to a certain standard of the same kind as itself, called the* **Unit.** *Thus the numerical value of an angle is its ratio to the angular unit; the numerical value of a line is the number of linear units (feet, &c.) which it contains.*

Mathematics are occupied with quantities, that is, with things that can be numbered, weighed, or measured; and each branch deals with a special kind. Thus Trigonometry is primarily occupied with calculations relating to lines and angles, and it is necessary to measure them in order that they may become subjects for computation. In these calculations the lines which enter are related to arcs of circles, and their numerical values are called *circular functions.* Trigonometry has for object the study of circular functions and their application to the resolution of triangles.

2. Two methods of measuring angles are employed in Trigonometry :—

1°. *The sexagesimal,* which is used in all practical applications, such as Geodesy, Navigation, Astronomy, &c.

2°. The *circular method,* employed in the various branches of *Analytical Mathematics.*

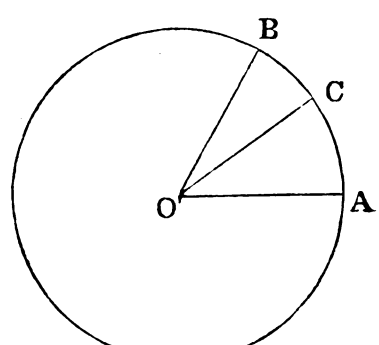
3. The Sexagesimal Method.—In this method a right angle is divided into 90 equal parts called *degrees;* a degree into 60 equal parts called *minutes;* a minute into 60 equal parts called *seconds.* Degrees and their subdivisions are denoted by the symbols °, ′, ″; thus 30° 22′ 25″ means 30 degrees, 22 minutes, 25 seconds. On the introduction of the metric system it had been proposed to divide an angle into 100 equal parts called *grades;* the grade into 100 equal parts called *minutes;* the minute into 100 equal parts called *seconds.* But this division, called the *centesimal,* has not been adopted.

4. The Circular Method.

Definition I.—*A unit circle* (in French "Cercle Trigonometrique") *is one whose radius is the unit of linear measure.*

Def. II.—*The unit of circular measure is the angle at the centre of the unit circle subtended by an arc of unit length.*

5. *In a unit circle any angle at the centre is measured by the same quantity which expresses the length of the subtended arc.*



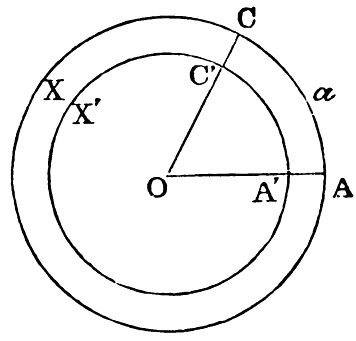
Demonstration.—Let AC be any arc, a its length, and A the corresponding central angle AOC. If AOB be the angular unit, since (Euc. VI. xxxiii.) angles at the centre are proportional to the arcs on which they stand, we have

$$AOC : AOB :: \text{arc } AC : \text{arc } AB,$$

or
$$A : 1 :: a : 1.$$

Hence
$$A = a. \tag{1}$$

6. *The circular measure of the angle subtended at the centre of a circle of radius r, by an arc whose length is a, is a/r.*



Dem.—Let X, X' be two circles whose radii are r and 1, respectively; AC, $A'C'$ two corresponding cs; then, since arcs subtended by equal angles at the centres of different circles are proportional to the radii of the circles (Euc. VI. xx., Ex. 12), we have, denoting the arc AC by a,

$$OA : OA' :: AC : A'C',$$

or

$$r : 1 :: a : A'C';$$

$$\therefore \quad A'C' = \frac{a}{r}; \tag{2}$$

but (§ 5) $A'C'$ is the circular measure of the angle AOC. Hence a/r is its circular measure.

7. Comparison of the Sexagesimal and Circular Methods.—If 2π denote the length of the circumference of the unit circle, then 2π is the number of angular units in four right angles; in other words, 2π is the circular measure of four right angles. Hence π is the circular measure of two right angles, and $\pi/2$ of one right angle.

π is evidently the ratio of the circumference of a circle to its diameter. It is incommensurable; that is, it cannot be denoted by the ratio of any two whole numbers. Its value is 3·1415926 approximately. We shall in a subsequent chapter show how to find this important constant.

Corollary 1.—The angular unit contains 206264·8 seconds $= 57^\circ\ 17'\ 44''\cdot 8$. For, if x denote the number of seconds in one unit of circular measure, $2\pi x$ is the number of seconds in 2π units, that is, in four right angles.

Hence

$$2\pi x = 360 . 60 . 60 = 1296000;$$

therefore

$$x = 206264\cdot 8, \text{ nearly.} \tag{3}$$

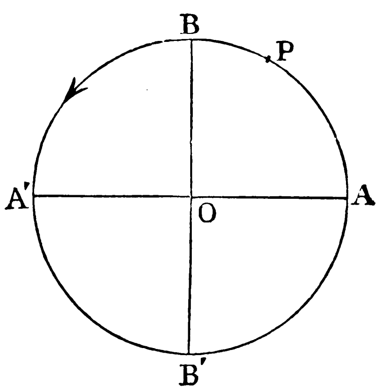
Cor. 2.—If a denote the length of an arc, and r the radius, the number of seconds in the corresponding angle is

$$(206264\cdot 8)\, a/r. \tag{4}$$

For a/r is (§ 6) the number of angular units in the arc, and there are 206264·8 seconds in each unit.

8. Rule of Signs of Descartes.—*Application to arcs of circles.*

Let a fixed point A on the unit circle be taken as the origin from which all arcs are measured; then if it be agreed that an arc described by a variable point P, moving in the direction indicated by the arrow, be considered as *positive*, an arc described by a point starting from A, and moving in the opposite direction, must be *negative*.



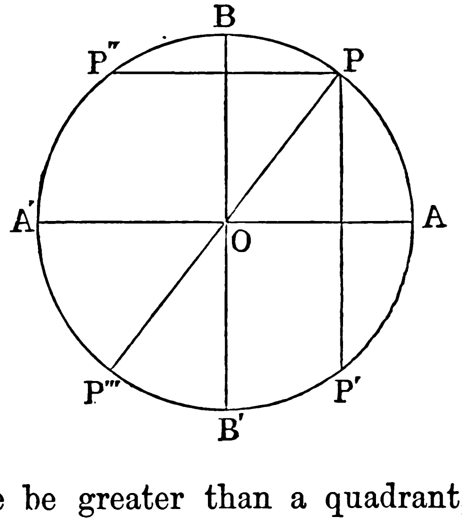
A little consideration will show that the distinction positive and negative in connection with the position of a point is absolutely necessary, and not merely a convention, as stated by some writers. All that is conventional is the direction fixed on as positive; but, whatever that be, the opposite must be negative.

9. General Expression for Arcs terminated in the same Point.—If the variable point starting from A, the origin of arcs, describe the entire circumference of the circle n times in either direction, and afterwards describe the arc AP, which we shall denote by θ, it is evident that the whole arc described from the commencement is $2n\pi + \theta$, where n is positive or negative, according to the direction of the moving point, in describing the n circumference. Hence we have the following theorems:—*If the arc AP be denoted by θ, and n be any integer, positive or negative, $2n\pi + \theta$ is the general expression for all arcs terminating in the point P.*

Cor.—The arc in Trigonometry may vary from $-\infty$ to $+\infty$.

Def. III.—*The diameter AA'* (fig., § 8) *passing through A, the origin of arcs, and the diameter BB' perpendicular to AA', divide the circle into four parts, which are called respectively AB the first quadrant, BA' the second, $A'B'$ the third, and $B'A$ the fourth quadrant.*

Def. IV.—*Two arcs whose sum is $\pi/2$ are said to be* **complements** *of each other, and two arcs whose sum is π* **supplements.**



Hence, if an are be greater than a quadrant, its *complement is negative*; and, if greater than a semicircle, its *supplement is negative.*

10. Observation.—Let P be any point on the unit circle; P', P'', P''' its reflections with respect to the diameters AA', BB'; then, if the arc AP be denoted by θ, the general expressions for arcs terminating in P, P', P'', P''', respectively, are—

$$2n\pi + \theta. \tag{5}$$
$$2n\pi - \theta. \tag{6}$$
$$(2n + 1)\pi - \theta. \tag{7}$$
$$(2n + 1)\pi + \theta. \tag{8}$$

Exercises.—I.

1. Find the number of degrees in the vertical angle of an isosceles triangle, each base angle of which is one-third of the vertical angle.

2. Find the circular measure of an angle of a regular octagon.

3. Find the circular measure of 11° 15′.

4. The earth's semidiameter, which is 3963 miles, subtends at the moon an angle of 57′ 3″·16; find the moon's distance.

5. It has been found by the transit of Venus, in 1882, that the earth's radius subtends at the sun an angle of 8″·82; find the sun's distance.

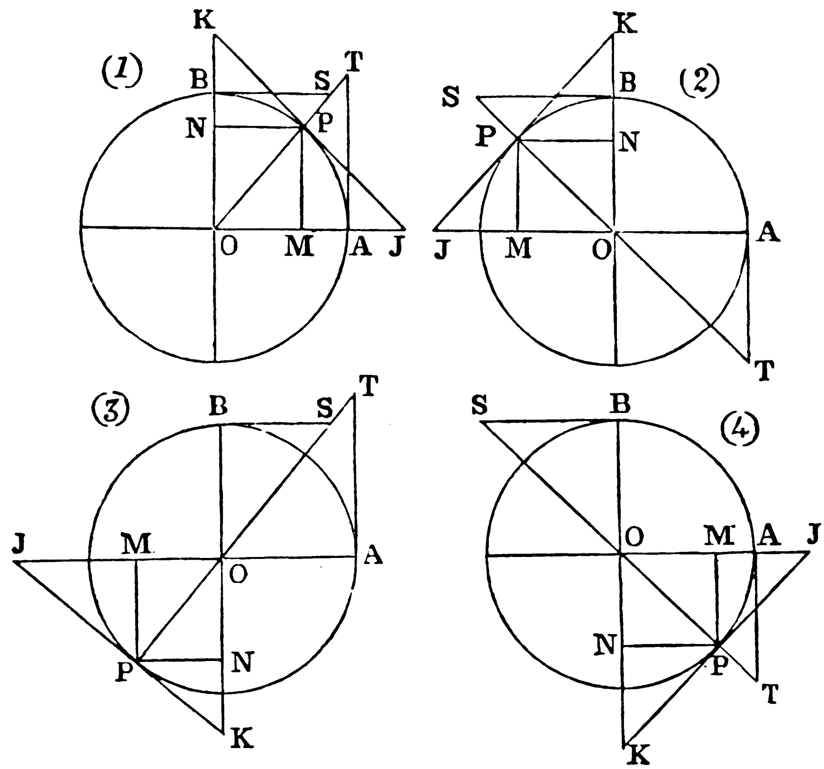
6. Struve has found, as the result of 241 meridian transits at Dorpat, that the vertical and horizontal semidiameters of the sun subtend at the earth angles of 960″·66, 961″·12, respectively; find these semidameters in miles.

7. The radius of the earth's orbit, which is 92,700,000 miles, subtends at α Centauri an angle of 75″; find the distance of α Centauri.

Section II.—Definitions of Circular Functions.

11. One magnitude is said to be a function of another, when to each value of the latter corresponds a determinate value of the former. For example, the area of a circle is a function of its radius. In this work, when we speak of a line we always mean its arithmetical value; in other words, the ratio of its length to the linear unit. In particular, it is in this sense we understand the lines (in French *lignes trigonométriques*) which represent the circular functions.

12. Let OA, OB (figures 1, 2, 3, 4) be two rectangular radii of the unit circle, P any point in the circumference. Join OP, and produce it to meet the tangents to the circle at A, B in the points T, S; at P draw a tangent, meeting OA, OB



produced in J, K; lastly, from P draw PM, PN perpendicular to OA, OB; let θ denote the arc AP; then we have the following definitions :—

MP is called the *sine* of the arc θ, contracted into sin θ.

OM	,,	*cosine*	,,	,,	cos θ.
AT	,,	*tangent*	,,	,,	tan θ.
BS	,,	*cotangent*	,,	,,	cot θ.
OJ	,,	*secant*	,,	,,	sec θ.
OK	,,	*cosecant*	,,	,,	cosec θ.

13. Since (§ 5) any arc is the measure of the corresponding central angle, the foregoing functions of the arc θ are also functions of the angle θ, and expressed by the same notation.

The following is the verbal enunciations of these definitions :—

1°. *The sine of an arc is the perpendicular drawn from its extremity on the diameter which passes through its origin.*

2°. *The tangent of an arc is the line drawn touching it at its origin, and terminated by the diameter passing through its extremity.*

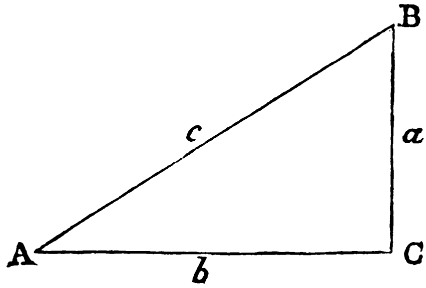
3°. *The secant of an arc is the intercept on the diameter passing through its origin between the centre and the tangent at its extremity.*

4°. *The cosine of an arc is the distance from the centre O to the foot of the sine.*

5°. *The cotangent of an arc is the tangent drawn through B, and comprised between B and the diameter drawn through the extremity of the arc.*

6°. *The cosecant of an arc is the portion of the diameter OB, intercepted between the centre and the tangent drawn through the extremity of the arc.*

14. Another Method of defining the Circular Functions.—



Let the angle BAC be denoted by A; then, if from any point

B in the line AB, BC be drawn perpendicular to AC, we have the following definitions:—

$$\sin A \ = \frac{BC}{AB}, \text{ or } BC = AB \sin A. \tag{9}$$

$$\cos A \ = \frac{AC}{AB}, \text{ or } AC = AB \cos A. \tag{10}$$

$$\tan A \ = \frac{BC}{AC}, \text{ or } BC = AC \tan A. \tag{11}$$

$$\cot A \ = \frac{AC}{BC}, \text{ or } AC = BC \cot A. \tag{12}$$

$$\sec A \ = \frac{AB}{AC}, \text{ or } AB = AC \sec A. \tag{13}$$

$$\operatorname{cosec} A = \frac{AB}{BC}, \text{ or } AB = BC \operatorname{cosec} A. \tag{14}$$

It is easy to see that this method of defining the circular functions of an angle is equivalent to the former when the angle is acute. For if the angle POM (§ 12, fig. 1) be equal to BAC, the triangles POM, BAC are equiangular. Hence $BC/AB = PM/OP$, but PM/OP is the arithmetical measure of PM, since OP is the linear unit. Hence BC/AB is equal to the arithmetical value of PM, and therefore both methods of defining the sine are equivalent, and the same may be shown for the other circular functions. The method by the right-angled triangle, however, has the defect of being inapplicable to any but acute angles without an embarrassing amount of explanation. The definitions in § 13 are free from the objections against the older circular definitions, which regarded the functions as lines, and have the advantage, as will be seen further on, of being easily expressed in a form that will suggest the remarkable analogy between them and the hyperbolic functions so important in recent Mathematical Physics.

15. Circular Functions of 45°, 30°, 60°.

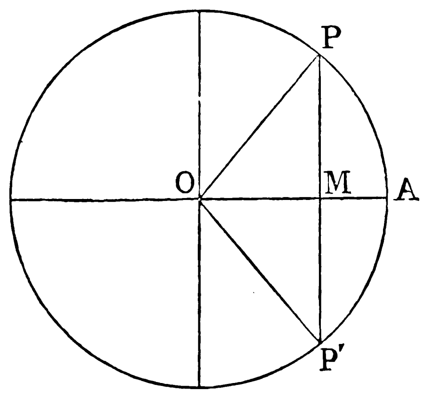
If we produce PM to P' we have: arc $PAP' = 2\theta$, $MP = \frac{1}{2}P'P$; then $\sin\theta = \frac{1}{2}$ chord 2θ. This relation gives immediately—

$$\sin 45^\circ = \tfrac{1}{2} \text{ chord } 90^\circ = \frac{1}{\sqrt{2}}. \tag{15}$$

$$\sin 30^\circ = \tfrac{1}{2} \text{ chord } 60^\circ = \tfrac{1}{2}. \tag{16}$$

$$\sin 60^\circ = \tfrac{1}{2} \text{ chord } 120^\circ = \frac{\sqrt{3}}{2}. \tag{17}$$

It is easy to verify by a figure that $\tan 45^\circ = 1$. (18)

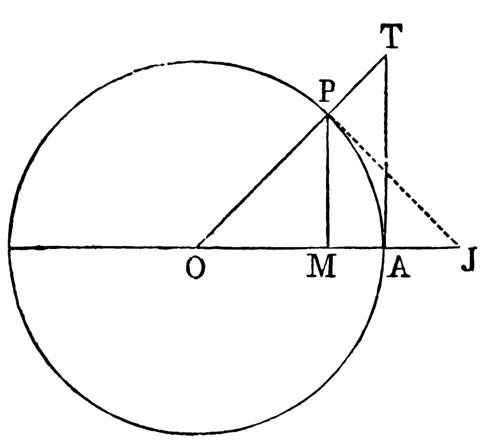


These values are useful to be remembered.

Exercises.—II.

1. Find by a construction an arc of the 1st quadrant, such that—

1°. $\sin\theta = \cos\theta$. 2°. $\sin\theta = 2\cos\theta$. 3°. $\tan\theta = 3\sin\theta$.



For 1°. We take $AT = OA$, and join OT.

For 2°. ,, $AT = 2OA$.

For 3°. ,, $OM = \frac{1}{3} OA$, and erect MP perpendicular.

2. Construct on the unit circle an arc of the 1st quadrant, such that—

1°. $\sin\theta = \frac{3}{4}$. 3°. $\tan\theta = \frac{7}{5}$. 5°. $\operatorname{cosec}\theta = 4$.

2°. $\cos\theta = \frac{5}{6}$. 4°. $\sec\theta = 3$. 7°. $\cot\theta = \frac{2}{3}$.

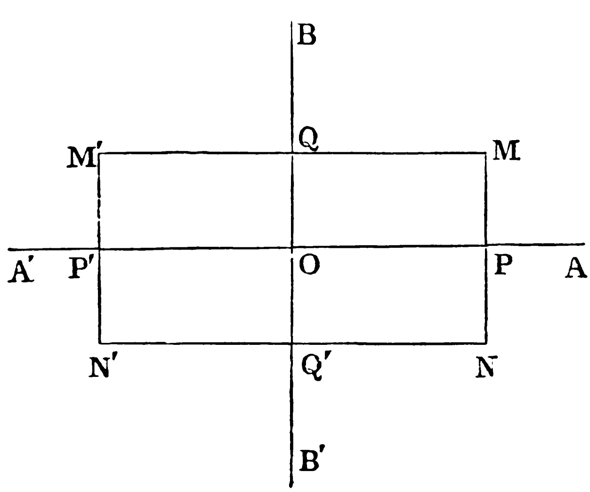
3. Construct an angle θ, such that $\sin\theta + \cos\theta = \frac{5}{4}$.

If we make $MJ = MP$, we should have $OJ = \frac{5}{4}$; then we take $OJ = \frac{5}{4}$, and make the angle $OJP = 45°$.

4. Construct an angle θ, such that $\cos\theta - \sin\theta = \frac{2}{3}$.

Section III.—Variations of the Circular Functions.

16. Descartes' Rule of Signs.—*Application to right lines.*—Let two perpendicular right lines AA', BB' intersect in the point O, which is the origin from which all lines are measured; then, every distance on AA' to the right of O, such as OP, is positive or +; those to the left, such as OP', are negative or –. Again, a line measured parallel to BB' is positive or + if it lie above AA', such as PM, $P'M'$; negative or – if it lie below AA', such as PN, $P'N'$.



17. Sine and Cosecant.—*These functions have always the same sign; which is positive in the* 1*st and* 2*nd quadrants, and negative in the* 3*rd and* 4*th.*

For in place of MP taking its equal ON for the sine, and OK being the cosecant, we see from the diagrams, § 12, that ON and OK are always measured in the same direction, and therefore have the same sign, which is + in 1st and 2nd quadrants, and − in 3rd and 4th. Again, suppose the movable point P to start from A, and describe the circumference $ABA'B'A$, the sine, which at first is 0, increases to unity, decreases to 0, then becomes negative, and varies in value from 0 to − 1, and from − 1 to 0. Thus, the maximum is + 1, and the minimum − 1.

When we consider a very small arc AP, its cosecant OK is very great, and as the point P moves towards A, the point K recedes indefinitely. This is what we mean when we say cosec $0 = \infty$. When the arc increases from 0 to $\pi/2$, the cosecant varies from ∞ to 1. In the second quadrant it varies from 1 to ∞; in the third, from $-\infty$ to -1; and in the 4th, from -1 to $-\infty$.

Cor.—If ϵ denotes an arc indefinitely small, we have

$$\operatorname{cosec}(\pi - \epsilon) = +\infty, \quad \operatorname{cosec}(\pi + \epsilon) = -\infty.$$

18. Cosine and Secant.—Since the cosine and secant are OM, OJ respectively, we see that both are positive in the 1st and 4th quadrants, and negative in the 2nd and 3rd. Also, as θ varies from 0 to 2π, $\cos\theta$ varies from 1 to 0, from 0 to − 1, from − 1 to 0, and from 0 to + 1. Hence the maximum is + 1; the minimum is − 1, and in changing sign it passes through zero. Sec θ varies from 1 to $+\infty$, from $-\infty$ to 1, from 1 to $-\infty$, from $+\infty$ to 1, and in changing sign passes through infinity.

19 Tangent and Cotangent.—These functions, being AT and BS, are positive in the 1st and 3rd quadrants, and negative in the 2nd and 4th. When θ varies from 0 to 2π, we see

that tan θ increases from 0 to $+\infty$; then from $-\infty$ to 0; then again from 0 to $+\infty$, and afterwards from $-\infty$ to 0. From these it follows that the tangent changes sign in passing through infinity. It will be seen that the variations of the cotangent correspond exactly to those of the tangent; but that while the tangent increases, the cotangent decreases.

These different variations are represented in the annexed Tables:—

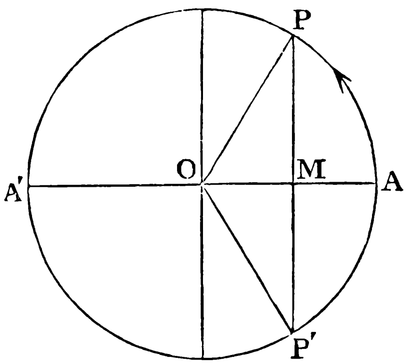
TABLE I.

Quadrant.	I.	II.	III.	IV.
Sin and cosec	+	+	−	−
Cos and sec	+	−	−	+
Tan and cot	+	−	+	−

TABLE II.

Quadrant.	I.	II.	III.	IV.
Sin varies from	0 to 1	1 to 0	0 to -1	-1 to 0
Cosec ,,	∞ ,, 1	1 ,, ∞	$-\infty$,, -1	-1 ,, $-\infty$
Cos ,,	1 ,, 0	0 ,, -1	-1 ,, 0	0 ,, 1
Sec ,,	1 ,, ∞	∞ ,, -1	-1 ,, $-\infty$	∞ ,, 1
Tan ,,	0 ,, ∞	$-\infty$,, 0	0 ,, ∞	$-\infty$,, 0
Cot ,,	∞ ,, 0	0 ,, $-\infty$	∞ ,, 0	0 ,, $-\infty$

20. Circular Functions of Negative Angles.—Let the arc AP on the unit circle be denoted by θ; then if the line PM be produced until it meet the circle again on the negative side of AA' in P', it is evident that the angle AOP' is equal in magnitude to AOP; but being measured in an opposite direc-



tion it has (§ 8) a contrary sign; hence it must be denoted by $-\theta$. Also, since MP, MP' are measured in opposite directions, they have contrary signs. Hence

$$MP' = -MP; \quad \text{but} \quad MP = \sin\theta,$$

and $$MP' = \sin(-\theta); \quad \therefore \sin(-\theta) = -\sin\theta. \tag{19}$$

Hence, if two arcs or angles be equal in magnitude, but have contrary signs, their sines are equal in magnitude and have contrary signs.

Again, the line OM is the cosine of the arc AP, and also the cosine of the arc AP'. Hence

$$\cos(-\theta) = \cos\theta. \tag{20}$$

Therefore, if two arcs differ only in sign, their cosines are equal. Similarly, it may be proved that

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta, \tag{21}$$

$$\tan(-\theta) = -\tan\theta, \tag{22}$$

$$\cot(-\theta) = -\cot\theta, \tag{23}$$

$$\sec(-\theta) = \sec\theta. \tag{24}$$

Section IV.—Circular Functions of Complemental Arcs, &c.

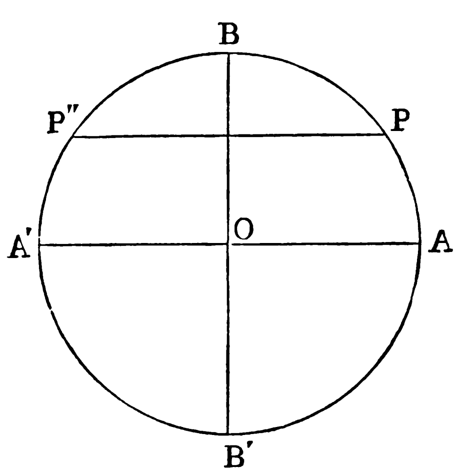
21. The complement of the arc AP is always BP, provided that the arc BP has for origin the point B, and that it be reckoned positive in the direction BA. From this it follows that the diameters AA', BB' change their rôles in the same manner as the tangents drawn through A and B; but that the positive directions of the trigonometrical lines remain the same Hence it follows that the cos, cot, cosec of an arc are respectively equal to the sin, tan, sec of its complement; therefore we can write

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right), \quad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right). \tag{25}$$

$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right), \quad \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right). \tag{26}$$

$$\operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right), \quad \sec\theta = \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right). \tag{27}$$

22. Supplemental Arcs.—If through P we draw PP'' parallel to AA', the arc $AP = P''A$. Hence the sum of the arcs AP,



AP'' is equal to the semicircle ABA'. Hence the arcs AP,

AP'' are supplements. And since the perpendiculars from P, P'' on the line AA' are equal, we have

$$\sin AP = \sin AP'', \quad \text{or} \quad \sin\theta = \sin(\pi - \theta).$$

In like manner we can compare their other trigonometrical functions: thus we find

$$\sin\theta = \sin(\pi - \theta), \quad \operatorname{cosec}\theta = \operatorname{cosec}(\pi - \theta), \qquad (28)$$

$$\cos\theta = -\cos(\pi - \theta), \quad \sec\theta = -\sec(\pi - \theta), \qquad (29)$$

$$\tan\theta = -\tan(\pi - \theta), \quad \cot\theta = -\cot(\pi - \theta). \qquad (30)$$

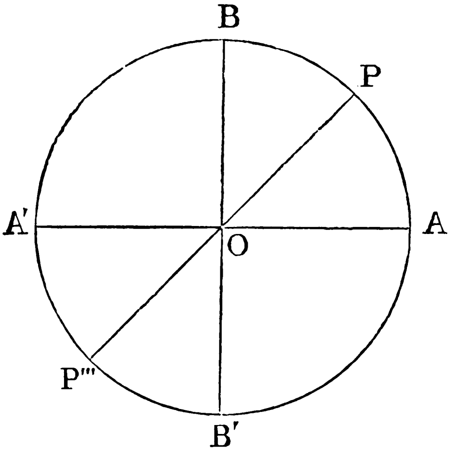
Thus the circular functions of supplemental arcs have contrary signs, except the sine and the cosecant, which are the same, but are equal in absolute values.

23. Arcs which differ by π.—The comparison of the circular functions of the arcs AP, AP''' shows that

$$\sin(\theta \pm \pi) = -\sin\theta, \quad \operatorname{cosec}(\theta \pm \pi) = -\operatorname{cosec}\theta. \qquad (31)$$

$$\cos(\theta \pm \pi) = -\cos\theta, \quad \cos(\theta \pm \pi) = -\sec\theta. \qquad (32)$$

$$\tan(\theta \pm \pi) = \tan\theta, \quad \cot(\theta \pm \pi) = \cot\theta. \qquad (33)$$



Hence, when we increase or diminish an arc by π, its circular functions retain their absolute values; but, except the tangent and cotangent, they change their signs.

24. Periodicity of the Circular Functions.—The circular functions of an arc depend only on its origin and its extremity.

Consequently the arcs comprised in the formula $(2n\pi + \theta)$ have the same circular functions. Hence

$$\sin(2n\pi + \theta) = \sin\theta), \qquad \cos(2n\pi + \theta) = \cos\theta. \qquad (34)$$

$$\operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta, \quad \sec(2n\pi + \theta) = \sec\theta. \qquad (35)$$

This property is expressed by saying that *the functions sin θ, cos θ, sec θ, cosec θ are periodic; and that the period is* 2π.

Again, since $\tan(\pi + \theta) = \tan\theta$, $\cot(\pi + \theta) = \cot\theta$; *the period for tan θ, cot θ is π.*

Cor. 1.—If we combine §§ 20–24, we can form the following Table:—

$$\sin(\overline{2n+1}\,.\,\pi + \theta) = -\sin\theta, \quad \cos(\overline{2n+1}\,.\,\pi + \theta) = -\cos\theta,$$
$$\tan(\overline{2n+1}\,.\,\pi + \theta) = \tan\theta. \qquad (36)$$

$$\sin(\overline{2n+1}\,.\,\pi - \theta) = \sin\theta, \quad \cos(\overline{2n+1}\,.\,\pi - \theta) = -\cos\theta,$$
$$\tan(\overline{2n+1}\,.\,\pi - \theta) = -\tan\theta. \qquad (37)$$

$$\sin(2n\pi - \theta) = -\sin\theta, \quad \cos(2n\pi - \theta) = \cos\theta,$$
$$\tan(2n\pi - \theta) = -\tan\theta. \qquad (38)$$

25. Reduction of an Arc to the 1st Quadrant.—We shall see that there are Tables in which are given the circular functions of all arcs included between 0 and $\pi/2$. If an arc is not comprised between these limits, we can find an arc in the 1st quadrant whose circular functions have the same absolute values.

If the arc lies between $\pi/2$ and π, we apply the rule of supplemental arcs, § 22. If it is comprised between π and 2π, we diminish it by π, by the rules of § 23, which reduces it to an arc between 0 and π. If it is greater than 2π, we commence by diminishing it by the greatest multiple of 2π

contained in it. If the arc is negative, we change its sign (§ 20).

EXAMPLES.

$$\cos 150^\circ = -\cos 30^\circ, \quad \tan 200^\circ = \tan 20^\circ, \quad \sin 240^\circ = -\sin 60^\circ,$$
$$\sec 300^\circ = -\sec(300^\circ - 180^\circ) = -\sec 120^\circ = \sec 60^\circ, \text{ \&c.}$$

Exercises.—III.

1. Reduce to the 1st quadrant $\sin 1000^\circ$, $\cos 1227^\circ$.

2. What relations are there between the circular functions of
$$\frac{\pi}{2} + \theta \text{ and } \theta?$$

3. What relations are there between the circular functions of
$$(4n+1)\frac{\pi}{2} - \theta \text{ and } \theta, \quad (4n+3)\frac{\pi}{2} - \theta \text{ and } \theta?$$

4. What relations are there between the circular functions of
$$(4n+1)\frac{\pi}{2} + \theta \text{ and } \theta, \quad (4n+3)\frac{\pi}{2} + \theta \text{ and } \theta?$$

5. Reduce to the 2nd quadrant
$$\sin(-10^\circ), \quad \cot 900^\circ, \quad \tan 800^\circ.$$

6. The quantities
$$\sin 2\theta, \quad \tan 5\theta, \quad \sec(a\theta + b), \quad \cos\frac{\theta}{3}, \quad \operatorname{cosec}\frac{3\theta + a}{4},$$
are periodic functions of θ. Indicate the period.

7. Simplify $\qquad a\cos 0^\circ - b\sec 180^\circ + c\sin 270^\circ$.

8. Simplify
$$5\sin 2\pi - 3\sin\frac{3\pi}{2} + 8\sin\pi - \sin\frac{\pi}{2} - \tan\frac{5\pi}{4} + \cot\frac{3\pi}{4}.$$

9–12. Simplify the following expressions:—

9. $\sin(\pi + \theta), \quad \cos(2\pi - \theta), \quad \cos\left(\frac{\pi}{2} + \theta\right), \quad \cos\left(\frac{3\pi}{2} + \theta\right), \quad \tan(\pi + \theta).$

10. $a\cos\left(\frac{\pi}{2} - \theta\right) + b\cos\left(\frac{\pi}{2} + \theta\right).$

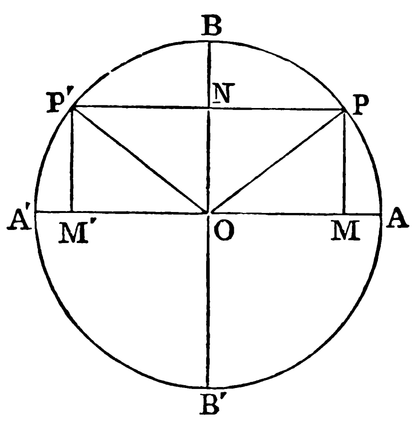
11. $(a+b)\tan(90^\circ - \theta) + (a-b)\cot(90^\circ + \theta).$

12. $\sin\theta \, . \tan(180^\circ + \theta) \div (\tan\theta \, . \cos(90^\circ - 0)).$

Section V.—Inverse Circular Functions.

26. We have seen in § 12, when an angle is given, or what is the same thing, the arc on the unit circle which is its measure, that we can construct its circular functions, and that each function has but one value corresponding to each value of the angle: the converse question is—Being given the value of the circular functions, to find the corresponding angles. We shall see in this case that to each value of a function there is an infinite number of corresponding angles. It will be sufficient to establish rules for the *sine*, *cosine*, and *tangent*, since the rules for these hold for the *cosecant*, *secant*, and *cotangent*.

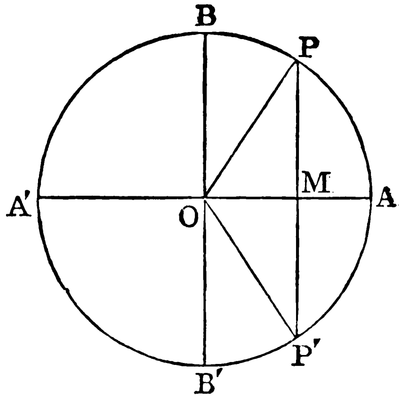
27. Sines.—*Let a be the sine of an arc. It is required to construct the arc.*



Solution.—Let O be the centre of the unit circle, A the origin of the arcs. Draw the diameters AA', BB' at right angles, and take on OB the line ON, whose numerical value is equal to a. Through N draw PP' parallel to AA'. Then, since MP, $M'P'$ are each equal to ON, the sines of the arcs AP, AP' are each equal to a; but the arcs AP, AP' are supplements (§ 22); and if one of them, AP, be denoted by α, the other, AP', will be $\pi - \alpha$. Again, since the sine of an angle is not altered when the angle is increased by $2n\pi$ (§ 24), it follows that if θ be the general value of the required angle, that

$$\theta = 2n\pi + \alpha, \quad \text{or} \quad \theta = (2n + 1)\pi - \alpha. \tag{39}$$

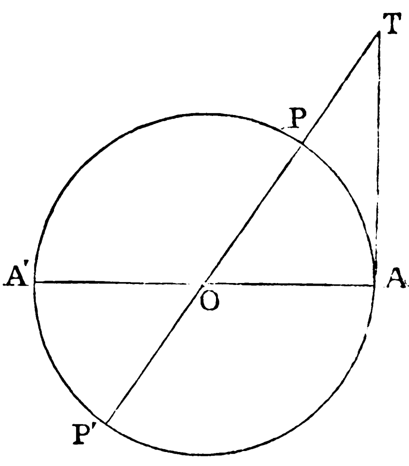
28. Cosines.—*It is required to find the general value of an angle whose cosine is a.*



Sol.—Let A be the origin; AA', BB' rectangular diameters. Take on OA the line OM, whose numerical value is equal to a. Through M draw PP' parallel to BB'; then it is evident the points P, P' are the extremities of all arcs whose cosines are equal to a. Now, if AP be denoted by α, AP' will be (§ 8) $-\alpha$. Hence, if θ be the general angle whose cosine is a, we have, n being any integer positive or negative,

$$\theta = 2n\pi \pm \alpha. \tag{40}$$

29. Tangents.—*It is required to find the general value of an arc whose tangent is a.*



Sol.—Let APA' be the unit circle, A the origin. Draw AT, touching the circle, and take AT such that its numerical value shall be equal to a. Join OT, cutting the circle in the points P, P'; then any arc terminating in either of these points will have its tangent equal to a. Hence, if $AP = \alpha$, and θ be the required arc,

$$\theta = n\pi + \alpha, \tag{41}$$

where n is any integer, positive or negative.

30. Def.—The following is the notation used for inverse functions:—

$$\sin^{-1} x, \quad \cos^{-1} y, \quad \tan^{-1} z, \quad \&c.$$

They are read thus:—*The arc whose sine is x, the arc whose cosine is y, the arc whose tangent is z*, &c. Thus, when we put $x = \sin\theta$, we can say conversely $\theta = \sin^{-1} x$. The inverse circular functions have an infinite number of values corresponding

to the same value of x. When it is convenient to remove the indeterminateness, we can do so by writing

$$\begin{array}{lcl} \sin^{-1}x & \text{between} & -\frac{\pi}{2} \text{ and } \frac{\pi}{2}; \\ \cos^{-1}x & ,, & 0 \text{ and } \pi; \\ \tan^{-1}x & ,, & -\frac{\pi}{2} \text{ and } \frac{\pi}{2}. \end{array}$$

The French method of writing inverse functions is nearly the same as they are read—thus: arc sin x, arc cos x, arc tan x.

Exercises.—IV.

1–4 on § 27. What is the general value of the angle whose sine is $\frac{1}{2}$?

2. What is the general value of the angle whose $\sin^2$ is $\frac{1}{2}$?

3. Write down all the values of θ which satisfy $\sin^2\theta = \sin^2\alpha$.

4. If two angles have the same sine, either their difference is an even multiple, or their sum an odd multiple, of π.

5–13 on § 28. Find the general value of θ which satisfies the following equations:—

5. $\sin 2\theta = \cos 3\theta$.

6. $\cos\theta + \cos 5\theta = 0$.

7. $\operatorname{cosec}\theta = \operatorname{cosec}\frac{\theta}{2}$.

8. $\cos\theta \,.\, \cos 5\theta + \cos 5\theta \,.\, \cos 7\theta = 0$.

9. $\sin 4\theta + \sin 8\theta = 0$.

10. Given $x + y = a$, $\sin 2x + \cos 3y = 0$; find x, y.

From the second equation,

$$\sin 2x = -\sin\left(\frac{\pi}{2} - 3y\right) = \sin\left(3y - \frac{\pi}{2}\right);$$

then $$2x - \left(3y - \frac{\pi}{2}\right) = 2n\pi, \quad \text{or} \quad 2x + 3y - \frac{\pi}{2} = (2n+1)\pi, \quad \&c.$$

11. Given $\cos x = \sin(\alpha - 2y)$, $\sin(\alpha - 3x) = \cos(x + 3y - \beta)$; find x, y.

12. Given $\sin(x - y) = \cos(x + y) = \frac{1}{2}$; find x, y.

13. If two arcs have the same cosine, either their sum or their difference must be an even multiple cf π.

14–24 on § 29. Find the general value of θ which satisfies the equations—

14. $\tan^2\theta = 1$. 15. $\tan^2\theta = \infty$. 16. $\tan^2\theta = 3$.

Solve the equations—

17. $\tan 2\theta = \tan(3\theta - \alpha)$. 18. $\tan(\theta - \alpha) + \cot(30 - \beta) = 0$.

19. $\tan\frac{1}{2}\theta = \cot(5\theta - \alpha)$.

20. $\begin{cases} mx + ny = \alpha; \\ \tan(mx - ny) = 1. \end{cases}$ 21. $\begin{cases} x + y = a; \\ \tan 2x + \tan 3y = 0. \end{cases}$

22. $\begin{cases} \sin 2x + \cos 3y = 0; \\ \tan 3x + \cot(4y - \alpha) = 0. \end{cases}$ 23. $\cos(4x - 3y) = \dfrac{1}{\sqrt{2}}\cot(x + y) = -1$.

24. If two arcs have the same tangent, their difference is a multiple of π.

CHAPTER II.

TRIGONOMETRICAL FORMULAE.

31. The formulae of Trigonometry can be classed into nine groups, as follows, to each of which we shall devote a Section :—

I. Formulae between functions of the same arc.
II. ,, for the addition of arcs.
III. ,, for multiple of arcs.
IV. ,, for submultiple of arcs.
V. ,, for transforming products into sums or differences.
VI. ,, for transforming sums or differences into products.
VII. ,, between inverse functions.
VIII. Trigonometric elimination.
IX. Trigonometric identities.

Section I.—Relation between the Circular Functions of an Angle.

32. Let θ denote the arc AP (figures 1, 2, 3, 4, § 12); then we have $OP = 1$, $\sin\theta = PM$, $\cos\theta = OM$, $\sec\theta = OJ$, $\operatorname{cosec}\theta = OK$, $\tan\theta = AT$, $\cot\theta = BS$. And since the angle OMP is right, $PM^2 + OM^2 = OP^2$.

Hence $$\sin^2\theta + \cos^2\theta = 1. \tag{42}$$

Also, since the triangles OPJ, OPK are right-angled, and PM, PN are perpendiculars, we have (Euc. VI. viii.)

$$ON \,.\, OK = OP^2, \quad OM \,.\, OJ = OP^2.$$

Hence $$\sin\theta \,.\, \operatorname{cosec}\theta = 1, \tag{43}$$

and $$\cos\theta \,.\, \sec\theta = 1. \tag{44}$$

Lastly, from the pairs of similar triangles TAO, PMO; SBO, PNO, we have

$$\frac{AT}{OA} = \frac{MP}{OP}; \quad \frac{BS}{OB} = \frac{NP}{ON};$$

that is, $$\tan\theta = \frac{\sin\theta}{\cos\theta}. \tag{45}$$

and $$\cot\theta = \frac{\cos\theta}{\sin\theta}. \tag{46}$$

From (45) and (46) we get $\tan\theta \,.\, \cot\theta = 1$. (47)

„ (43) we have $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$. (48)

„ (44) „ $\sec\theta = \dfrac{1}{\cos\theta}$. (49)

„ (47) „ $\cot\theta = \dfrac{1}{\tan\theta}$. (50)

From the three last formulae we see that *cosec θ, sec θ, cot θ, are the reciprocals of sin θ, cos θ, tan θ.*

Again, since the triangles OPJ, OAT are equal in every respect,

$$OJ^2 = OT^2 = OA^2 + AT^2; \;\therefore\; \sec^2\theta = 1 + \tan^2\theta. \tag{51}$$

In like manner, from the triangles OPK, OBS, we have

$$OK^2 = OS^2 = OB^2 + BS^2; \;\therefore\; \operatorname{cosec}^2\theta = 1 + \cot^2\theta. \tag{52}$$

33. Use of the preceding Formulae.—When a circular function is given, the extremity of the arc is known (but it can have two positions, §§ 27, 28, 29), and the other functions are therefore known, for any five can be expressed in terms of the sixth. For example, if $\sin\theta$ be given, we have

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}, \qquad \tan\theta = \frac{\pm\sin\theta}{\sqrt{1-\sin^2\theta}},$$

$$\cot\theta = \frac{\pm\sqrt{1-\sin^2\theta}}{\sin\theta}, \quad \sec\theta = \frac{\pm 1}{\sqrt{1-\sin^2\theta}}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}. \tag{53}$$

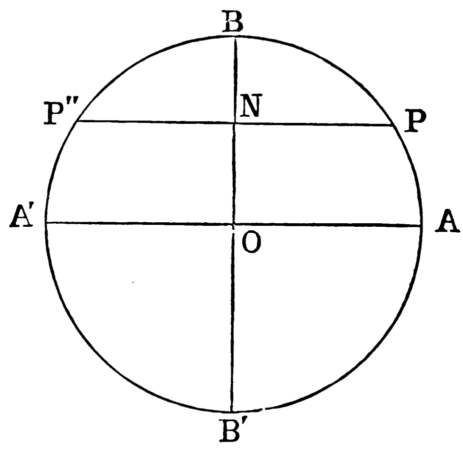
In like manner, if $\tan\theta$ be given, we have—

$$\sin\theta = \frac{\tan\theta}{\pm\sqrt{1+\tan^2\theta}}, \quad \cos\theta = \frac{\pm 1}{\sqrt{1+\tan^2\theta}}, \quad \cot\theta = \frac{1}{\tan\theta},$$

$$\sec\theta = \pm\sqrt{1+\tan^2\theta}, \quad \operatorname{cosec}\theta = \frac{\pm\sqrt{1+\tan^2\theta}}{\tan\theta}, \qquad (54)$$

and so on for the other functions.

The double sign of the radical admits of an easy explanation. For example, if $\sin\theta = ON$, θ is terminated in P or P''; and $\cos\theta = +\sqrt{1-\sin^2\theta}$ in the first case, and $= -\sqrt{1-\sin^2\theta}$ in the second.



34. The formulae of § 32 are often employed to verify formulae, or to transform a given formula.

EXAMPLES.

1. Verify the formula $\dfrac{1+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta} = \dfrac{1-\tan\theta}{1+\tan\theta}$.

Replacing $\tan\theta$ by $\dfrac{\sin\theta}{\cos\theta}$,

$$\frac{1-2\sin\theta\cos\theta}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)} = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta};$$

suppressing the factor, $\cos\theta+\sin\theta$;

we get $1-2\sin\theta\cos\theta = (\cos\theta-\sin\theta)^2$,

which is exact.

2. Admitting that the arcs x, y satisfy the equation

$$a \sin x \sin y + b \cos x \cos y = c;$$

find the relation between $\tan x$, $\tan y$.

Dividing by $\cos x$, $\cos y$, and using the values of $\sec x$, $\sec y$, from equation (54), we get

$$a \tan x \tan y + b = c\sqrt{(1 + \tan^2 x)(1 + \tan^2 y)};$$

and clearing of radicals, we get

$$(c^2 - a^2) \tan^2 x \tan^2 y + c^2 (\tan^2 x + \tan^2 y) - 2ab \tan x \tan y + (c^2 - b^2) = 0.$$

35. Trigonometric Equations.—In Trigonometry an unknown arc x is determined very often, by means of an equation which contains several trigonometric functions of x, $2x$, $x/2$, &c. In order to solve these equations, it is necessary to transform them so that they will contain only a single circular function of an unknown arc. This may be the arc x itself, or a function of x, such as $x/2$, $2x$, $x + a$. . . . Having found the value of the function by solving the equation, the trigonometrical tables, as we shall see further on, will enable us to determine x.

EXAMPLES.

1. Given $\qquad 2 \sec x = \tan x + \cot x$; find x.

$$\frac{2}{\cos x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x};$$

$$\therefore \quad 2 \sin x = \sin^2 x + \cos^2 x = 1, \quad \sin x = \tfrac{1}{2}.$$

First Solution.— $\qquad x = \frac{\pi}{6}.$

General Solution.— $\qquad \left\{ \begin{array}{l} x = 2n\pi + \frac{\pi}{6}, \\ x = (2n + 1)\pi - \frac{\pi}{6}. \end{array} \right.$

2. $\qquad a \sin^2 x + b \cos^2 x = c, \quad a \sin^2 x + b(1 - \sin^2 x) = c,$

$$\sin x = \pm \sqrt{\frac{c - b}{a - b}} = \pm \sqrt{\frac{b - c}{b - a}}.$$

In order that $\sin x$ may be real and < 1, it is easy to see that either

$$a > c > b, \quad \text{or} \quad a < c < b.$$

Exercises.—V.

1. Given $\sin\theta = \frac{3}{5}$, find $\cos\theta$.

2. ,, $\cos\theta = \frac{8}{17}$, ,, $\sin\theta$.

3. ,, $\sin\theta = \frac{12}{13}$, ,, $\tan\theta$.

4. ,, $\tan\theta = \frac{3}{4}$, ,, $\sec\theta$.

5. ,, $\sin 10'' = \cdot 0000484813681$, prove that

$$2(1 - \cos 10'') = \cdot 0000000023504. \tag{55}$$

[This result is useful in calculating Tables of the circular functions.]

6. In a right-angled triangle ABC, having given the angle C right, if the sides opposite the angles A, B, C be denoted by a, b, c, respectively, then being given

1°. $a = \sqrt{m^2 + n^2}$, $b = \sqrt{2mn}$, calculate $\cos A$.

2°. $a = 2mn$, $b = m^2 - n^2$, ,, $\sin A$.

3°. $a = \sqrt{m^2 + mn}$, $c = m + n$, ,, $\tan A$.

4°. $a = m^2 + mn$, $c = mn + n^2$, ,, $\cot A$.

Prove the following identities:—

7. $\dfrac{1}{\cot\theta} \cdot \dfrac{1}{\sec\theta} = \sin\theta$.

8. $\tan\theta \,.\, \sin\theta + \cos\theta = \sec\theta$.

9. $(\sin\theta + \cos\theta) \div (\sec\theta + \operatorname{cosec}\theta) = \sin\theta\cos\theta$.

10. $(\sin\theta + \cos\theta)^2 \div \sin\theta\cos\theta = (1 + \tan\theta)(1 + \cot\theta)$.

11. $(\sec\theta + \operatorname{cosec}\theta)^2 = (1 + \tan\theta)^2 + (1 + \cot\theta)^2$.

12. If the sides of a right-angled triangle be in A, P, prove that the sines of its acute angles are $\frac{3}{5}$, $\frac{4}{5}$, respectively.

13. Find the circular functions of an angle of 45°.

Since the sine of an angle is equal to the cosine of its complement,

$$\sin^2 45° - \cos^2 45° = 0,$$

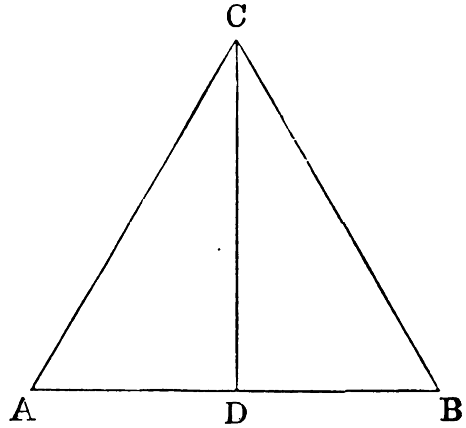
and

$$\sin^2 45° + \cos^2 45° = 1.$$

Hence $2\sin^2 45° = 1$, and $\sin 45° = \dfrac{1}{\sqrt{2}}$; (compare equation (15)):

$$\therefore \quad \cos 45° = \frac{1}{\sqrt{2}}, \quad \tan 45° = \cot 45° = 1, \quad \sec 45° = \operatorname{cosec} 45° = \sqrt{2}. \tag{56}$$

14. Find the values of the circular functions of 60°.



Let ABC be an equilateral triangle, CD a perpendicular on AB; then the angle is 60°. And since the triangle is isosceles,

$$AD = DB; \quad \therefore\ AD = \tfrac{1}{2}AC.$$

Hence $\quad AD \div AC = \frac{1}{2};$

$\therefore \quad \cos CAD$ or $60° = \frac{1}{2}$. (57)

Hence $\quad \sin 60° = \frac{\sqrt{3}}{2},$

$$\tan 60° = \sqrt{3}, \quad \cot 60° = \frac{1}{\sqrt{3}},$$

$$\sec 60° = 2, \quad \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

Cor.—Since 30° is the complement of 60°, $\sin 30° = \cos 60° = \frac{1}{2}$. (58)

Hence $\quad \cos 30° = \frac{\sqrt{3}}{2}, \quad \tan 30° = \frac{1}{\sqrt{3}}, \quad \cot 30° = \sqrt{3},$

$$\sec 30° = \frac{2}{\sqrt{3}}, \quad \operatorname{cosec} 30° = \sqrt{2}. \qquad (59)$$

15. Prove $\quad \cot^2 A - \cot^2 B = \dfrac{\cos^2 A - \cos^2 B}{\sin^2 A \,.\, \sin^2 B}.$ (60)

16. Prove $(\sin 60° - \sin 45°)(\cos 30° + \cos 45°) = \sin^2 30°.$

17. ,, $(\cos 30° - \sin 30°)(\sin 60° + \cos 60°) = \sin^2 45°.$

18. ,, $(\sin 45° + \sin 30°) \div (\sin 45° - \sin 30°) = \sec 45° + \tan 45°)^2.$

19. ,, $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta.$

20. ,, $\tan\theta + \cot\theta = \sec\theta \,.\, \operatorname{cosec}\theta.$

21. ,, $(\cot\theta + \operatorname{cosec}\theta)^2 = \dfrac{1+\cos\theta}{1-\cos\theta}.$

22. ,, $\operatorname{cosec}^6\theta - \cot^6\theta = 1 + 3\operatorname{cosec}^2\theta\cot^2\theta.$

23. ,, $\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta) = \sec\theta + \operatorname{cosec}\theta.$

24. ,, $1 + 2(\sin^6\theta + \cos^6\theta) = 3(\sin^4\theta + \cos^4\theta).$

25. If $\cos\theta = \dfrac{\sin^2 A}{\cos^2 A} + 1$, prove $\sec\theta = \cos^2 A.$

Prove the following, by means of a construction:—

26. That $2 \sin\theta$ is greater than $\sin 2\theta$.

27. „ $(\tan\theta - \sin\theta)^2 + (1 - \cos\theta)^2 = (\sec\theta - 1)^2$.

28. „ $(\operatorname{cosec}\theta - \sec\theta)^2 = (1 - \tan\theta)^2 + (\cot\theta - 1)^2$.

29. That the diameter of the unit circle which bisects the first quadrant divides the circle into two semicircles, such that $\sin\theta - \cos\theta$ is positive for all arcs terminating in any point of one of them, and negative for all arcs terminating in any point of the other.

30. That the diameter which bisects the second quadrant is similarly related to the expression $\sin\theta + \cos\theta$.

31–36. Find the values of the functions in the following equations:—

31. $\cos^2\theta = \frac{3}{2}\sin\theta$.

32. $\tan\theta + \cot\theta = \frac{10}{3}$.

33. $\tan\theta = 2\sin\theta$.

34. $\sec^4\theta + 6 = 5\sec^2\theta$.

35. $9\sin^2\theta + 27\sin\theta = 10$.

36. $\cos^2\theta - 2\sin\theta + \frac{1}{4} = 0$.

37–44. Find the general value of θ in the equations—

37. $2\sin\theta = \tan\theta$.

38. $6\tan^2\theta - 4\sin^2\theta = 1$.

39. $\tan\theta - \cot\theta = \sqrt{2}$.

40. $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2}$.

41. $\operatorname{cosec}\theta \,.\, \cot\theta = 2\sqrt{3}$.

42. $\dfrac{\sin\theta}{1 + \cos\theta} = 2 - \cot\theta$.

43. $\sec\theta = \dfrac{2}{\sqrt{3}}\tan\theta$.

44. $(\cot\theta - \tan\theta)^2 = 4\dfrac{(2 - \sqrt{3})}{(2 + \sqrt{3})}$.

45–47. Transform into expressions which will contain only $\cos\alpha$.

45. $\dfrac{\sin^2\alpha}{\cos\alpha} + \dfrac{\tan\alpha}{\cot\alpha}$.

46. $\cot^2\alpha + \tan^2\alpha - \sin^2\alpha$.

47. $\sin\alpha\cos\alpha\tan\alpha \,.\, \cot\alpha$.

48. Express $\cos\theta + \dfrac{\tan\theta \,.\, \cos\theta}{\sin\theta} + \dfrac{\sin^2\theta}{\cos\theta}$ in terms of $\sec\theta$.

49. Express $\sec\theta + \cos\theta + \tan\theta$ in terms of $\operatorname{cosec}\theta$.

50. Solve the equations

$$\sin x + \sin y = \tfrac{1}{4}. \quad (1)$$

$$\cos x \cos y = \tfrac{3}{4}. \quad (2)$$

Equation (2) gives

$$(1 - \sin^2 x)(1 - \sin^2 y) = \tfrac{9}{16},$$

$$1 - \sin^2 x - \sin^2 y + \sin^2 x \sin^2 y = \tfrac{9}{16}. \quad (3)$$

Add (3) to the square of (1), and we get

$$(1 + \sin x \sin y)^2 = \tfrac{10}{16}, \quad \sin x \sin y = \frac{-4 + \sqrt{10}}{4}. \quad (4)$$

The sign $-$ before $\sqrt{10}$ must be rejected, because $(\sin x \sin y)^2 < 1$.

From (1) and (4) it follows that $\sin x$, $\sin y$, are the roots of the equation

$$z^2 - \tfrac{1}{4} z + \frac{-4 + \sqrt{10}}{4} = 0. \quad (5)$$

Then
$$\sin x = \tfrac{1}{8}\left(1 \pm \sqrt{65 - 16\sqrt{10}}\right),$$
$$\sin y = \tfrac{1}{8}\left(1 \mp \sqrt{65 - 16\sqrt{10}}\right).$$

These values are real, and in absolute value less than 1. Let α and $-\beta$ be the absolute values of the smallest arcs which correspond to them.

Then
$$x = 2n\pi + \alpha, \quad \text{or} \quad (2n + 1)\pi - \alpha,$$
$$y = 2n\pi - \beta, \quad \text{or} \quad (2n + 1)\pi + \beta.$$

Since equation (2) was squared, it is necessary to verify that the preceding values are such that $\cos x \cos y = +\frac{3}{4}$, and not $-\frac{3}{4}$. This requires that α and β are comprised between 0 and $\dfrac{\pi}{2}$.

51. Given $\sin x + \cos x = a$, $\sin^3 x + \cos^3 x = b$; prove $a^3 - 3a + 2b = 0$.

52. Prove $(\sec\theta - \cos\theta) \div (\operatorname{cosec}\theta - \sin\theta) = \tan^3\theta$.

53–65. Solve the equations

53. $2\sin^2\theta - 3\sin\theta + a = 0$. (Discuss what are the possible values of a, that $\sin\theta$ may be real, and comprised between $+1$ and -1.)

54. $\tan^2\theta - 2\tan\theta\tan\alpha + 1 = 0$. (Discuss, &c.)

55. $2\tan^2\theta - 3\tan\theta + 2a = 0$. (Discuss, &c.)

56. $2\sin^2\theta - 3\sin\theta\cos\theta + \cos^2\theta = 0$.

57. $\sin\alpha\tan^2 x - 2\cos\alpha\tan x + 1 = 0$. (Discuss, &c.)

58. $(\sin x + \tan x) \div (\tan x - \sin x) = 5 + 2\sqrt{5}$.

59. $a\cos^2 x + (2a^2 - a + 1)\sin x - 3a + 1 = 0$.

60. $(1 + m)\sin^2 x - 3m\sin^3 x + m\cos^2 x(1 - 3\sin x) = 0$.

61. $a\sin^2\theta + b\tan^2\theta = c$.

62. $\sin x = a\sin y$, $\tan x = b\tan y$. Find x, y.

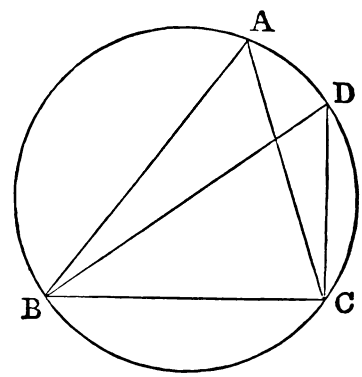
63. $\sin x = a\sec y$, $\cos x = b\operatorname{cosec} y$. „ „

64. $\tan x + \tan y = a$, $\cot x + \cot y = b$. „ „

65. $a\tan x + b\cot y = 1$, $a'\cot x + b'\tan y = 1$. „ „

Section II.—Formulae for the Addition of Arcs.

36. Lemma.—*In a triangle* (ABC) *any side* (AB), *divided by the diameter of the circumcircle, is equal to the sine of the opposite angle* (C).



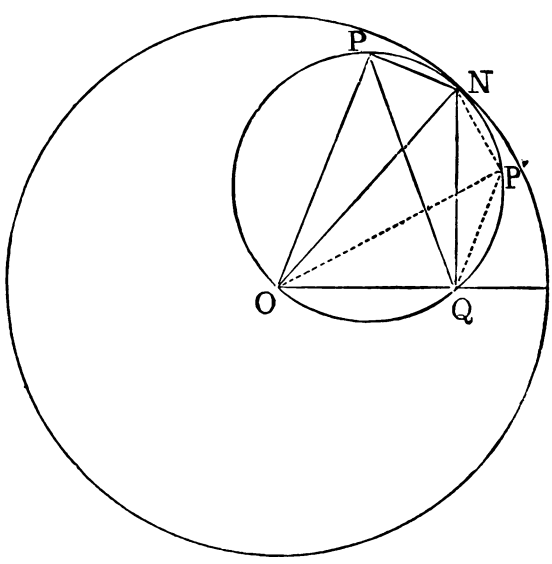
Dem.—Draw the diameter BD. Join AD. Then, because BD is a diameter, the angle BAD is right; $\therefore AB/BD = \sin BDA$—that is (Euc. III. xxi.) $= \sin BCA$. Hence, denoting the diameter by δ; the angles of the triangle by A, B, C; and the opposite sides by a, b, c, we have

$$a/\delta = \sin A. \tag{61}$$

Cor.—In any triangle $a : b :: \sin A : \sin B$.

For $$a/\delta = \sin A, \quad b/\delta = \sin B.$$

37. *The sine of the sum of two angles* (α, β) *is equal to the sine of the first, multiplied into the cosine of the second, plus the cosine of the first into the sine of the second.*



Dem.—Let ON be a radius of the unit circle; and let the

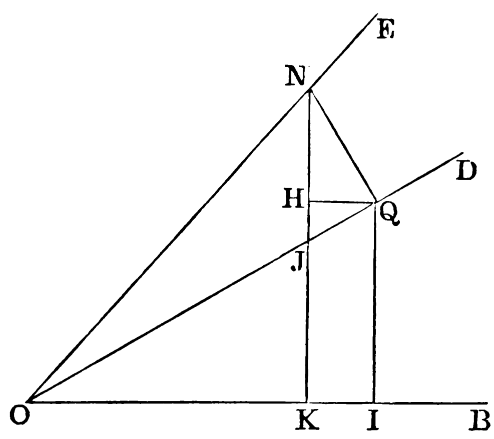
angles QON, NOP be denoted by α, β, respectively; let fall the perpendiculars NP, NQ. Join PQ. Then, since the points O, P, N, Q are concyclic, by Ptolemy's theorem (Euc. p. 232), we have $PQ \,.\, ON = QN \,.\, OP + OQ \,.\, NP$; but ON is unity. Hence, by the lemma, $PQ = \sin POQ = \sin(\alpha + \beta)$; also, $QN = \sin\alpha$, $NP = \sin\beta$, $OQ = \cos\alpha$, $OP = \cos\beta$;

$$\therefore \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \qquad (62)$$

38. If the line OP lie between ON and OQ, such as OP', we have, by Ptolemy's theorem, $P'Q \,.\, ON = QN \,.\, OP' - OQ \,.\, NP'$

Hence $$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \qquad (63)$$

39. The two formulae (62) (63) may be proved otherwise. Thus:—Let the two angles BOD, DOE be denoted by α, β; then BOE will be $\alpha + \beta$. In OE take any point N. Draw NK perpendicular to OB, NQ to OD, and QI to OB; also draw QH parallel to OB. Now, since the triangles OKJ, JQN are right-angled, and have the angles OJK, NJQ equal

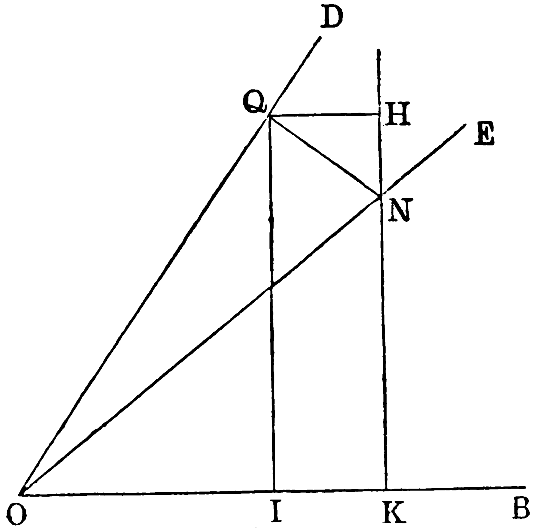


(Euc. I. xv.), the angle JOK is equal to JNQ. Hence JNQ is equal to α. Again,

$$\sin(\alpha + \beta) = \frac{KN}{ON} = \frac{KH + HN}{ON} = \frac{IQ}{ON} + \frac{HN}{ON}$$

$$= \frac{IQ}{OQ} \cdot \frac{OQ}{ON} + \frac{HN}{QN} \cdot \frac{QN}{ON} = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

the same as before. Again, let OE lie between OB and OD; then the angle BOE will be $\alpha - \beta$, and we have



$$\sin(\alpha - \beta) = \frac{KN}{ON} = \frac{KH - NH}{ON} = \frac{IQ}{ON} - \frac{NH}{ON}$$

$$= \frac{IQ}{OQ} \cdot \frac{OQ}{ON} - \frac{HN}{QN} \cdot \frac{QN}{ON} = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \text{ as in § 38.}$$

40. *The cosine of the sum of two angles* (α, β) *is equal to the product of their cosines, minus the product of their sines.*

Dem.—$\cos(\alpha + \beta) = \frac{OK}{ON} = \frac{OI - KI}{ON} = \frac{OI}{ON} - \frac{HQ}{ON}$ (1st fig. § 39)

$$= \frac{OI}{OQ} \cdot \frac{OQ}{ON} - \frac{HQ}{QN} \cdot \frac{QN}{ON} = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \quad (64)$$

41. *The cosine of the difference of two angles* (α, β) *is equal to the product of their cosines, plus the product of their sines.*

Dem.—$\cos(\alpha - \beta) = \frac{OK}{ON} = \frac{OI + IK}{ON} = \frac{OI}{ON} + \frac{QH}{ON}$ (2nd fig. § 39)

$$= \frac{OI}{OQ} \cdot \frac{OQ}{ON} + \frac{QH}{QN} \cdot \frac{QN}{ON} = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \quad (65)$$

42. The formulae (62)–(65) are called the *four fundamental*

formulae of Trigonometry. On account of their importance, several proofs of them have been given by mathematicians. The following, due to Mr. W. Nichols (*Messenger of Mathematics,* 1864), is one of the most elegant:—

Make the angle $BAC = \alpha$, $CAD = \beta$ (figs. (1) (2)). Draw BC perpendicular to AC, meeting AD in D, and BE perpendicular to AD, meeting AC in F. Now in fig. (1) we have—

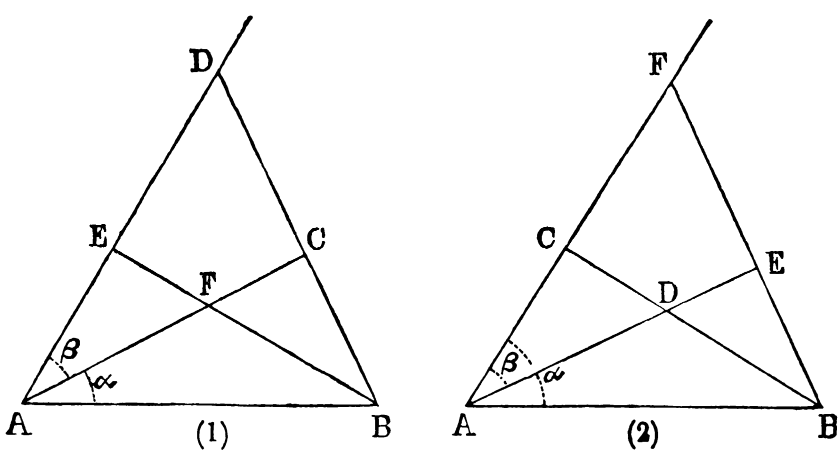
$$\sin(\alpha+\beta) = \frac{BE\,.\,AD}{AB\,.\,AD} = \frac{BD\,.\,AC}{AB\,.\,AD} = \frac{(BC+CD)AC}{AB\,.\,AD}$$

$$= \frac{BC\,.\,AC}{AB\,.\,AD} + \frac{AC\,.\,CD}{AB\,.\,AD} = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In fig. (2) we have

$$\sin(\alpha-\beta) = \frac{BE\,.\,AD}{AB\,.\,AD} = \frac{BD\,.\,AC}{AB\,.\,AD} = \frac{(BC-CD)AC}{AB\,.\,AD}$$

$$= \frac{BC\,.\,AC}{AB\,.\,AD} - \frac{AC\,.\,CD}{AB\,.\,AD} = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$



Similarly, from figures (1) and (2), we have

$$\cos(\alpha\pm\beta) = \frac{AE\,.\,AF}{AB\,.\,AF} = \frac{(AC\mp FC)AE}{AB\,.\,AF}$$

$$= \frac{AC\,.\,AE}{AB\,.\,AF} \mp \frac{BC\,.\,EF}{AB\,.\,AF} = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

The equality of the rectangles $BE \cdot AD$, $BD \cdot AC$, assumed in the foregoing proof, follows from each being equal to twice the area of the triangle ABD; and the equality $FC \cdot AE$, $BC \cdot EF$ from the similar triangles AEF, BCF.

43. General Demonstration of the Fundamental Formulae.— The demonstrations of §§ 37–42 suppose that α, β are positive arcs whose sum is less than $\pi/2$. But since these formulae serve to establish all others in Trigonometry, it is necessary to demonstrate them in all their generality.

Lemma I.—The projection $A'B'$ of a line AB upon an axis XY is equal in magnitude and sign to the right line AB multiplied by the cosine of the angle between the positive directions of $A'B'$ and AB.

Dem.—Let α be the acute angle between AB and XY, θ the angle between the positive directions of AB and XY. Draw AZ parallel to XY, cutting BB' in C. We have, independent of sign,

$$\cos \alpha = \frac{AC}{AB}.$$

I say, taking signs into account, that

$$\cos \theta = \frac{AC}{AB}.$$

There are four cases to be considered:—

1°. AB and AC are $+$; then $\theta = \alpha$. (Fig. 1.)

2°. AB is $+$, and AC $-$; then $\theta = \pi - \alpha$;

$\dfrac{AC}{AB}$ is $-$, and $\cos\theta = -\cos\alpha$. (Fig. 2.)

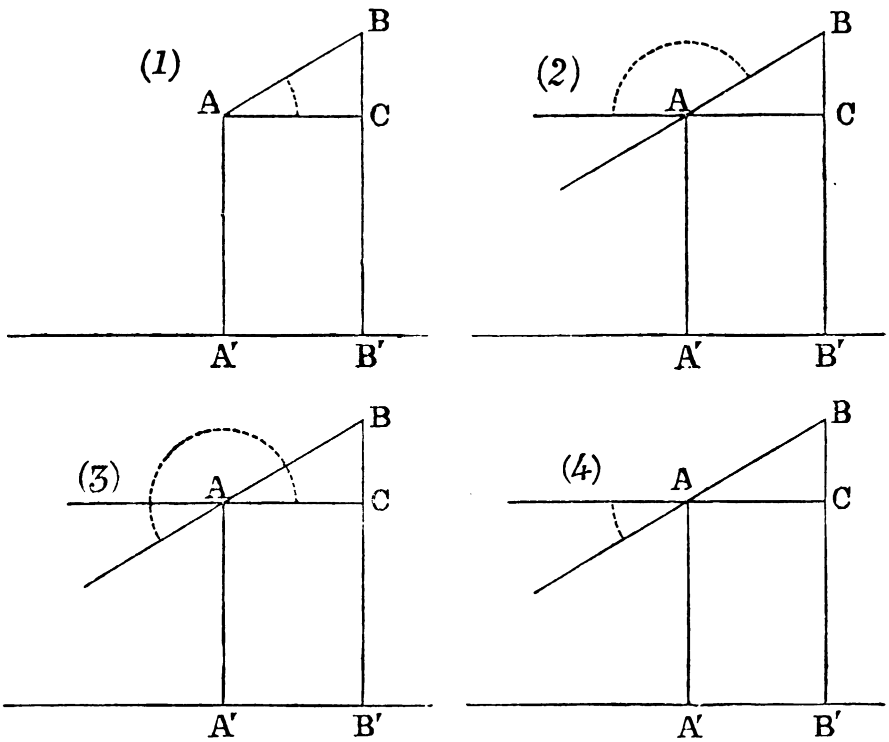
3°. AB is $-$, and AC $+$; then $\theta = \pi + \alpha$;

$\dfrac{AC}{AB}$ is $-$, and $\cos\theta = -\cos\alpha$. (Fig. 3.)

4°. AB and AC are $-$; then $\theta = \alpha$;

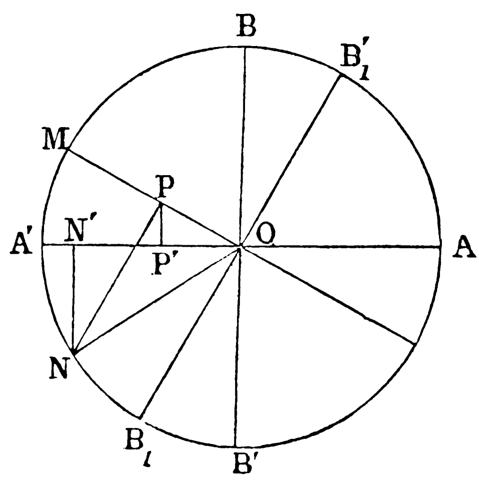
$\dfrac{AC}{AB}$ is $+$, and $\cos\theta = \cos\alpha$. (Fig. 4.)

Lemma II.—The sum of the projections of a closed contour $ABCD \ldots KA$ upon an axis is null.



For, if A', B', C', &c. . . . be the projections of its summits, we have
$$A'B' + B'C' + \ldots K'A' = 0.$$
Hence the proposition is proved.

44. Universal Proof.—Let $AM = \alpha$, $MN = \beta$. Draw OB_1



perpendicular to OM, and in the positive direction from OM. Join OM, ON. Draw NP perpendicular to OM, and let P', N' be the projections of the points P, N on the axis AA'; then we have (Lemma II.),

$$OP' + P'N' + N'O = 0;$$

but $\quad OP' = OP\cos(OM \hat{\ } OA) = \cos\beta\cos\alpha$

(Lemma I., OM is the + direction of OP),

$$P'N' = PN\cos(OB_1 \hat{\ } OA) = \sin\beta\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\beta\sin\alpha,$$

$$N'O = -ON\cos(ON \hat{\ } OA) = -\cos(\alpha + \beta);$$

$\therefore \quad \cos\beta\cos\alpha - \sin\beta\sin\alpha - \cos(\alpha + \beta) = 0;$

$\therefore \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

This formula being true for all values of α and β, we can replace β by $-\beta$. We get

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Replacing α by $\frac{\pi}{2} - \alpha$, we get

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Replacing β by $-\beta$, we get

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.*$$

45. If we divide (62) by (64), and reduce the right-hand side by dividing numerator and denominator by $\cos\alpha\cos\beta$, we get

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \qquad (66)$$

In like manner, from (63) and (65), we get

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \qquad (67)$$

These results may be proved geometrically. (See figs., § 42.)

* The foregoing proof is, with slight alteration, the same as that given in "Leçons de Trigonométrie," by MM. Briot and Bouquet. See also "Cours de Trigonométrie," by MM. Vacquant and Lépinay.

We have $\tan(\alpha \pm \beta) = \frac{BE}{AE}$ = by similar triangles to

$$\frac{BD}{AF} = \frac{BC \pm CD}{AC \mp CF} = \left(\frac{BC}{AC} \pm \frac{CD}{AC}\right) \div \left(1 \mp \frac{CF}{CB} \cdot \frac{CB}{AC}\right)$$

$$= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}.$$

46. Formuale for $\sin(\alpha + \beta + \gamma)$, $\cos(\alpha + \beta + \gamma)$,

$$\sin(\alpha + \beta + \gamma) = \sin(\alpha + \beta)\cos\gamma + \cos(\alpha + \beta)\sin\gamma$$

(equation (62).)

And, substituting for

$$\sin(\alpha + \beta), \quad \cos(\alpha + \beta),$$

their values from (62), (64), we get

$$\sin(\alpha + \beta + \gamma) = \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\gamma\cos\alpha$$
$$+ \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma,$$

which may be written

$$\sin(\alpha + \beta + \gamma) = \Sigma\sin\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma. \quad (68)$$

Similarly,

$$\cos(\alpha + \beta + \gamma) = \cos\alpha\cos\beta\cos\gamma - \Sigma\cos\alpha\sin\beta\sin\gamma. \quad (69)$$

Remark.—We may in the same manner find

$$\sin(\alpha + \beta + \gamma + \delta), \quad \cos(\alpha + \beta + \gamma + \delta) \ldots$$

47. Development of tan $(\alpha + \beta + \gamma \ldots)$

We shall first find the expression for

$$\tan(\alpha + \beta + \gamma).$$

If we divide (68) by (69), and reduce the right-hand side by dividing both numerator and denominator by

$$\cos\alpha\cos\beta\cos\gamma,$$

we get

$$\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}. \quad (70)$$

Cor. 1.—If $\alpha + \beta + \gamma = n\pi$, then $\tan(\alpha + \beta + \gamma = 0$.
Hence the numerator on the left-hand side of (70) must vanish, and therefore

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma, \qquad (71)$$

or

$$\cot\beta\cot\gamma + \cot\gamma\cot\alpha + \cot\alpha\cot\beta = 1. \qquad (72)$$

Cor. 2.—If $(\alpha + \beta + \gamma) = (4n \pm 1)\frac{\pi}{2}$,
the denominator on the right side of (70) must vanish;

$$\therefore \quad \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1, \qquad (73)$$

or

$$\cot\alpha + \cot\beta + \cot\gamma = \cot\alpha\cot\beta\cot\gamma. \qquad (74)$$

48. Let there be n arcs $\alpha, \beta, \gamma \dots \lambda$; and, denoting by s_p the sum of the products of their tangents p by p, so that

$$s_1 = \tan\alpha + \tan\beta + \tan\gamma, \text{ \&c.},$$
$$s_2 = \tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma + \dots \text{ \&c.}$$

Then
$$\tan(\alpha + \beta + \gamma \dots \lambda) = \frac{s_1 - s_3 + s_5 - \text{\&c.}}{1 - s_2 + s_4 - \text{\&c.}} \qquad (75)$$

In fact, if the formula be true for n arcs $\alpha, \beta \dots \lambda$, it will be true for $(n + 1)$ arcs $\alpha, \beta \dots \gamma, \mu$; for

$$\tan(\alpha + \beta \dots + \lambda + \mu) = \frac{\tan(\alpha + \beta \dots + \lambda) + \tan\mu}{1 - \tan(\alpha + \beta) \dots \lambda)\tan\mu};$$

and, substituting in this the value of $\tan(\alpha + \beta \dots + \lambda)$, we get

$$\frac{(s_1 - s_3 + s_5 - \dots) + \tan\mu\,(1 - s_2 + s_4 - \text{\&c.}}{1 - s_2 + s_4 - \text{\&c.} - \tan\mu\,(s_1 - s_3 + s_5 - \text{\&c.}} = \tan(\alpha + \beta \dots + \lambda + \mu);$$

and, denoting by s'_p the sum of the products of the tangents of the $(n + 1)$ arcs $\alpha, \beta \dots \lambda, \mu$, taken p by p, we have

$$s_1 + \tan\mu = s'_1, \quad s_3 + s_2\tan\mu = s'_3, \text{ \&c.}$$

Hence
$$\tan(\alpha + \beta \dots \lambda + \mu) = \frac{s'_1 - s'_3 - s'_5 - \text{\&c.}}{1 - s'_2 + s'_4 - \text{\&c.}};$$

but the formula is true for three arcs, therefore for four, &c. And therefore it is universally true.

Exercises.—VI.

1. Given $\sin\alpha = \frac{3}{5}$, $\cos\beta = \frac{9}{41}$; find $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$.

2. ,, $\sin\alpha = \frac{8}{17}$, $\sin\beta = \frac{196}{371}$; ,, $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$.

3. Prove $\sin\alpha \pm \cos\alpha = \sin(\alpha \pm 45^\circ)\sqrt{2}$. (76)

4. ,, $\cos\alpha \pm \sin\alpha = \cos(\alpha \mp 45^\circ)\sqrt{2}$. (77)

5. ,, $\sin\alpha = \cos\beta\sin(\alpha+\beta) - \sin\beta\cos(\alpha+\beta)$.

6. ,, $\sin\alpha = \cos\beta\sin(\alpha-\beta) + \sin\beta\cos(\alpha-\beta)$.

7. ,, $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$. (78)

8. ,, $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$. (79)

9. ,, $\sin 3^\circ = \dfrac{\sin^2 2^\circ - \sin^2 1^\circ}{\sin 1^\circ}$. (80)

10. ,, $\sin(30^\circ+\alpha) + \sin(30^\circ-\alpha) = \cos\alpha$. (81)

11. ,, $\cos(60^\circ+\alpha) + \cos(60^\circ-\alpha) = \cos\alpha$. (82)

12. ,, $\sin(60^\circ+\alpha) - \sin(60^\circ-\alpha) = \sin\alpha$. (83)

[The results (81), (82), (83) are useful in the construction of Tables.]

13. Prove $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$. (84)

14. ,, $\tan\alpha - \tan\beta = \dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$. (85)

15. ,, $\tan(\alpha+45^\circ) = \dfrac{1+\tan\alpha}{1-\tan\alpha}$. (86)

16. ,, $\tan(\alpha-45^\circ) = \dfrac{1-\tan\alpha}{1+\tan\alpha}$. (87)

17. ,, $\cot\beta \pm \cot\alpha = \dfrac{\sin(\alpha\pm\beta)}{\sin\alpha \,.\, \sin\beta}$. (88)

18. If $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, prove $(\alpha+\beta) = \dfrac{\pi}{4}$. (89)

19. ,, $\tan\alpha = \frac{5}{6}$, $\tan\beta = \frac{1}{11}$, ,, $\alpha+\beta = \dfrac{\pi}{4}$. (90)

20. ,, $\tan(\alpha\pm\beta) = \dfrac{\cot\beta \pm \cot\alpha}{\cot\alpha\cot\beta \mp 1}$. (91)

21. Prove $\cot(\alpha \pm \beta) = \dfrac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$. (92

22. ,, $\sin(\alpha+\beta) + \cos(\alpha-\beta) = (\sin\alpha + \cos\alpha)(\sin\beta + \cos\beta)$. (93)

23. ,, $\sin^2(\alpha+\beta) = \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$. (94)

24. ,, $\dfrac{\sin(\alpha+\beta)+\cos(\alpha-\beta)}{\sin(\alpha-\beta)+\cos(\alpha+\beta)} = \dfrac{\sin\beta+\cos\beta}{\cos\beta-\sin\beta}$. (95)

25. ,, $\tan(\alpha+\beta)\tan(\alpha-\beta) = \dfrac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha-\sin^2\beta}$.

26. ,, $\tan(\alpha+\beta) = \dfrac{\sin^2\alpha-\sin^2\beta}{\sin\alpha\cos\alpha-\sin\beta\cos\beta} = \dfrac{\sin\alpha\cos\alpha+\sin\beta\cos\beta}{\cos^2\alpha-\sin^2\beta}$.

27. ,, $\dfrac{\tan\alpha}{1-\cot 2\alpha\tan\alpha} = \sin 2\alpha$.

28. ,, $\dfrac{\cot 2\alpha}{\cot 2\alpha+\tan\alpha} = \cos 2\alpha$.

29. ,, $\dfrac{\tan 2\alpha+\tan\alpha}{\tan 2\alpha-\tan\alpha} = \dfrac{\sin 3\alpha}{\sin\alpha}$.

30–40. Solve the following equations:—

30. $\sin(x+a) = a\sin x + b\cos x$.

31. $\sin x + \sin y = a, \quad \cos x + \cos y = b$.

32. $\tan(x+45^\circ) = a\tan x$.

33. $\sin x = a\sin(45^\circ - x)$.

34. $\sin x\sin(\alpha - x) = m\cos^2 x$.

35. $\tan(60^\circ + x)\tan(60^\circ - x) = m$.

36. $3\sin x = 2\sin(60^\circ - x)$.

37. $\tan x\tan(x+45^\circ) = m$.

38. $\tan(45^\circ + x) + \tan(45^\circ - x = m$.

39. $\tan\alpha\tan x = \tan^2(\alpha+x) - \tan^2(\alpha-x)$.

40. $\sin x/\sin y = 2, \quad \cos(x+y)/\cos(x-y) = \frac{2}{3}$.

41. Prove $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} + \dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma} + \dfrac{\sin(\gamma-\alpha)}{\sin\gamma\sin\alpha} = 0$.

Section III.—Formulae for Multiple Arcs.

49. *Values of sin* $2a$, *cos* $2a$, *tan* $2a$.

If we make $a = \beta$ in formulae (62), (64), (66), we get

$$\sin 2a = 2 \sin a \cos a. \tag{96}$$

$$\cos 2a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a. \tag{97}$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}. \tag{98}$$

Remark.—If we write $\frac{1}{2}a$ in place of a, the formulae just written become

$$\sin a = 2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a. \tag{99}$$

$$\cos a = \cos^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} a = 2 \cos^2 \tfrac{1}{2} a - 1 = 1 - 2 \sin^2 \tfrac{1}{2} a. \tag{100}$$

$$\tan a = \frac{2 \tan \frac{1}{2} a}{1 - \tan^2 \frac{1}{2} a}. \tag{101}$$

From the last formula we get

$$2 \cot a = \cot \tfrac{1}{2} a - \tan \tfrac{1}{2} a, \tag{102}$$

and from (100) $$1 + \cos a = 2 \cos^2 \tfrac{1}{2} a \tag{103}$$

and $$1 - \cos a = 2 \sin^2 \tfrac{1}{2} a. \tag{104}$$

In (99) change a into $(a + \beta)$ and we get,

$$\sin (a + \beta) = 2 \sin \tfrac{1}{2} (a + \beta) \cos \tfrac{1}{2} (a + \beta). \tag{105}$$

50. *Values of sin* $3a$, *cos* $3a$, *tan* $3a$.

If we make $a = \beta = \gamma$ in formulae (68), (69), (70), we get

$$\sin 3a = 3 \sin a - 4 \sin^3 a. \tag{106}$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a. \tag{107}$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}. \tag{108}$$

These values may also be obtained by making $\beta = 2a$ in the formulae for

$$\sin (a + \beta), \quad \cos (a + \beta), \quad \tan (a + \beta).$$

51. *Values of* $\sin n\alpha$, $\cos n\alpha$, $\tan n\alpha$.

If the values of $\sin(n-1)\alpha$, $\cos(n-1)\alpha$, $\tan(n-1)\alpha$ were known in functions of $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, the values of $\sin n\alpha$, $\cos n\alpha$, $\tan n\alpha$ could be found. For, putting $\beta = (n-1)\alpha$ in formulae (62), (64), (66), we get

$$\sin n\alpha = \sin(n-1)\alpha\cos\alpha + \cos(n-1)\alpha\sin\alpha. \qquad (109)$$

$$\cos n\alpha = \cos(n-1)\alpha\cos\alpha - \sin(n-1)\alpha\sin\alpha. \qquad (110)$$

$$\tan n\alpha = \frac{\tan(n-1)\alpha + \tan\alpha}{1-\tan(n-1)\alpha\tan\alpha}. \qquad (111)$$

If we make $\alpha = \beta = \gamma \ldots = \lambda$ in (75) we get

$$\tan n\alpha = \frac{n\tan\alpha - \frac{n\,.\,n-1\,.\,n-2}{\lfloor 3}\tan^3\alpha + \&c.}{1 - \frac{n\,.\,n-1}{\lfloor 2}\tan^2\alpha + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{\lfloor 4}\tan^4\alpha}, \ \&c. \qquad (112)$$

52. *Values of* $\sin 2\alpha$, $\cos 2\alpha$, *in functions of* $\tan\alpha$.

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}. \qquad (113)$$

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}. \qquad (114)$$

These are proved by substituting on the right for $\tan\alpha$ the value $\frac{\sin\alpha}{\cos\alpha}$. They are remarkable as being rational.

Exercises.—VII.

1. Find $\cos 60°$.

$\sin 120°$ is equal to $\sin 60°$, because $60°$ is the supplement of $120°$;

$$\therefore\ 2\sin 60°\,.\,\cos 60° = \sin 60°.$$

Hence $$\cos 60° = \tfrac{1}{2}.$$

2. Find $\sin 30°$.

Since $60°$ is the complement of $30°$, $\sin 60° = \cos 30°$

$$\therefore\ 2\sin 30°\cos 30° = \cos 30°$$

$$\therefore\ \sin 30° = \tfrac{1}{2}.$$

3. Find sin 18°.

Putting $\alpha = 18°$, we have, since $\sin 36° = \cos 54°$,

$$\sin 2\alpha = \cos 3\alpha,$$

or $$2 \sin \alpha \cos \alpha = 4 \cos^3 \alpha - 3 \cos \alpha.$$

Hence $$2 \sin \alpha = 4 \cos^2 \alpha - 3 = 4(1 - \sin^2 \alpha) - 3;$$

$$\therefore \quad 4 \sin^2 \alpha + 2 \sin \alpha = 1.$$

Hence $$\sin \alpha = \sin 18° = \frac{\sqrt{5} - 1}{4}. \qquad (115)$$

Cor.— $$\cos 18° = \frac{\sqrt{10 + 2\sqrt{5}}}{4}. \qquad (116)$$

4. Prove $\dfrac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} = \sec 2\alpha - \tan 2\alpha.$

5. ,, $\cos 2\alpha = 2 \cos \left(\frac{\pi}{4} - \alpha\right) \cos \left(\frac{\pi}{4} + \alpha\right).$

6. ,, $\cos^8 \alpha - \sin^8 \alpha = \cos 2\alpha \, (1 - \frac{1}{2} \sin^2 2\alpha).$

7. ,, $\cos^6 \alpha + \sin^6 \alpha = 1 - \frac{3}{4} \sin^2 2\alpha.$

8. ,, $\sin 4\alpha = 4 \sin \alpha \cos \alpha - 8 \sin^3 \alpha \cos \alpha.$

9. ,, $\sin 5\alpha = 5 \sin \alpha - 20 \sin^3 \alpha + 16 \sin^5 \alpha.$

10. ,, $\sin 6\alpha = 6 \sin \alpha \cos \alpha - 32 \sin^3 \alpha \cos \alpha + 32 \sin^5 \alpha \cos \alpha.$

11. ,, $\cos 4\alpha = 1 - 8 \cos^2 \alpha + 8 \cos^4 \alpha.$

12. ,, $\cos 5\alpha = 5 \cos \alpha - 20 \cos^3 \alpha + 16 \cos^5 \alpha.$

13. ,, $-\cos 6\alpha = 1 - 18 \cos^2 \alpha + 48 \cos^4 \alpha - 32 \cos^6 \alpha.$

14. ,, $4 \sin 3\alpha \cos^3 \alpha + 4 \cos 3\alpha \sin^3 \alpha = 3 \sin 4\alpha.$

15. ,, $\tan 2\alpha = \dfrac{\tan \alpha}{1 - \tan \alpha} + \dfrac{\tan \alpha}{1 + \tan \alpha} = \dfrac{1}{1 - \tan \alpha} - \dfrac{1}{1 + \tan \alpha}.$

16. ,, $2 \tan 2\alpha = \cot (45° - \alpha) - \tan (45° - \alpha).$

17. ,, $\sin 3\alpha = 4 \sin \alpha \, (\sin^2 60° - \sin^2 \alpha) = 4 \sin \alpha \, . \sin 60° - \alpha) \sin (60° + \alpha).$

18. ,, $\cos 3\alpha = 4 \cos \alpha \, (\cos^2 60° - \cos^2 \alpha) = 4 \cos \alpha \, . \cos (60° - \alpha) \cos (60° + \alpha).$

19. ,, $\tan 3\alpha = \tan \alpha \, . \tan (60° - \alpha) \tan (60° + \alpha).$

20. ,, $\sin 20° \, . \sin 40° \, . \sin 60° \, . \sin 80° = \frac{3}{16}.$

21. ,, $\cos 20° \, . \cos 40° \, . \cos 60° \, . \cos 80° = \frac{1}{16}.$

22. ,, $\tan 20° \, . \tan 40° \, . \tan 60° \, . \tan 80° = 3.$

Solve the following equations :—

23. $\sin 2x = a \cos x$.
24. $3 \sin x = \sin 2x + \sin 3x$.
25. $\tan x + \tan 2x = \tan 3x$.
26. $\sin x = a \sin^2 x$.
27. $\cos 2x = 4 \sin x$.
28. $\cot x - \tan x = \sin x + \cos x$.
29. $\cos 2x + 3 \sin x = a$.
30. $\dfrac{1 - \tan x}{1 + \tan x} = 2 \cos 2x$.
31. $4 \cos 2x + 3 \cos x = 1$.
32. $\tan x + 2 \cot 2x = \sin x \,(1 + \tan x /2 \tan x$.
33. $\tan x - \cot x = \cot 2x - 4$.
34. $\sin 2x = 6 \sin^2 x - 8 \cos^2 x$.
35. $\sin^4 x + \cos^4 x = \sin 2x$.
36. $\tan^2 2x - \tan^2 x = 2 \tan 2x \tan a$.
37. $\tan x \,.\, \tan 3x = 2$.
38. $\cot 2x = a\, \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.
39. $\sin x + \cos x = a$.
40. $4 \sin 5x = \operatorname{cosec} x$.

41. Prove

$$x - \sin x = 4 \left\{ \sin^3 \frac{x}{3} + 3 \sin^3 \frac{x}{3^2} + 3^2 \sin^3 \frac{x}{3^3}, \text{ \&c., to inf.} \right\}. \quad (117)$$

DEM.—In (106), put $a = \dfrac{x}{3}$, and we get

$$3 \sin \frac{x}{3} - \sin x = 4 \sin^3 \frac{x}{3}.$$

In this equation, change x into $\dfrac{x}{3}$ and multiply by 3. Treat the result in the same manner, and so on; thus we get—

$$3^2 \sin \frac{x}{3^2} - 3 \sin \frac{x}{3} = 4 \,.\, 3 \sin^3 \frac{x}{3^2},$$

$$3^3 \sin \frac{x}{3^3} - 3^2 \sin \frac{x}{3^2} = 4 \,.\, 3^2 \sin^3 \frac{x}{3^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$3^n \sin \frac{x}{3^n} - 3^{n-1} \sin \frac{3}{3^{n-1}} = 4 \,.\, 3^{n-1} \sin^3 \frac{x}{3^n}.$$

Hence

$$3^n \sin \frac{x}{3^n} - \sin x = 4 \left\{ \sin^3 \frac{x}{3} + 3 \sin^3 \frac{x}{3^2} + 3^2 \sin^3 \frac{x}{3^3} \ldots 3^{n-1} \sin^3 \frac{x}{3^n} \right\}.$$

Now, if n be indefinitely large, $\dfrac{x}{3^n}$ will be indefinitely small, and therefore

$$\sin \frac{}{3^n} = \frac{x}{3^n}. \quad \text{Hence } 3^n \sin \frac{x}{3^n} = x;$$

$$x - \sin x = 4 \left\{ \sin^3 \frac{x}{3} + 3 \sin^3 \frac{x}{3^2} + 3^2 \sin^3 \frac{x}{3^3}, \text{ \&c., to inf.} \right\}.$$

42. Prove $\sin\frac{\pi}{15}=\frac{1}{8}\left\{\sqrt{(10+2\sqrt{5})}-\sqrt{3}(\sqrt{5}-1)\right\}.$ (118)

43. ,, $\cos\frac{\pi}{15}=\frac{1}{8}\left\{\sqrt{3}\sqrt{(10+2\sqrt{5})}+(\sqrt{5}-1)\right\}.$ (119)

[Make use of the relation $\pi/15=\pi/6-\pi/10$.]

44. ,, $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}},\quad \cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$ (120)

45. ,, $\tan\pi/12=2-\sqrt{3},\quad \cot\pi/12=2+\sqrt{3}.$ (121)

46. ,, $\sin\frac{3\pi}{20}=\frac{1}{4}\left\{\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})}\right\}.$ (122)

47. ,, $\cos\frac{3\pi}{20}=\frac{1}{4}\left\{\sqrt{(5+\sqrt{5})}+\sqrt{(3-\sqrt{5})}\right\}.$ (123)

48. ,, $\sin\frac{\pi}{60}=\frac{1}{8}\left\{(\sqrt{3}+1)\sqrt{(3-\sqrt{5})}-(\sqrt{3}-1)\sqrt{(5+\sqrt{5})}\right\}.$ (124)

49. ,, $\cos\frac{\pi}{60}=\frac{1}{8}\left\{(\sqrt{3}+1)\sqrt{(5+\sqrt{5})}+(\sqrt{3}-1)\sqrt{(3-\sqrt{5})}\right\}.$ (125)

[Make use of the relations $\frac{3\pi}{20}=\frac{\pi}{4}-\frac{\pi}{10};\quad \frac{\pi}{60}=\frac{\pi}{12}-\frac{\pi}{15}.$]

50. ,, $\sin x > x - x^3/6.$ (126)

Since the sin of an arc is less than the arc,

$$\sin\frac{x}{3}<\frac{x}{3}\quad \sin\frac{x}{3^2}<\frac{x}{3^2},\ \&c.$$

Hence, from equation (117), we have

$$x-\sin x<4\left\{\frac{x^3}{3^3}+\frac{x^3}{3^5}+\frac{x^3}{3^7},\ \&c.\right\};\ \text{that is,} <\frac{x^3}{6}.$$

Hence $$\sin x > x - x^3/6.$$

51. Prove $\cos x < 1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}.$ (127)

$$\cos x = 1-2\sin^2\frac{x}{2};\ \text{therefore}\ \cos x > 1-2\left(\frac{x}{2}-\frac{x^3}{48}\right)^2;$$

that is, less than $1-\frac{x^2}{\lfloor 2}+\frac{x^4}{24}-\frac{x^6}{1152}.$ Hence, &c.

The results in (126), (127) are very important in the construction of Tables.

SECTION IV.—FORMULAE FOR SUBMULTIPLE ARCS.

53. *If the cosine of an angle be given, the sine and the cosine of its half are each two-valued.*

DEM.— $\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta = \cos\theta$, equation (100),

and $\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta = 1$. ,, (42).

Hence
$$\cos \tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}. \qquad (128)$$

$$\sin \tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}. \qquad (129)$$

Hence, if $\cos\theta$ *be given,* $\sin\frac{1}{2}\theta$ *and* $\cos\frac{1}{2}\theta$ *have each two given values.*

Or thus:—If the given value of $\cos\theta$ be a, and if α be a determinate angle, whose cosine is a, then the general value of § 28 is given by the equation

$$\theta = 2n\pi \pm \alpha.$$

Hence
$$\tfrac{1}{2}\theta = n\pi \pm \tfrac{1}{2}\alpha;$$

$$\therefore \sin\tfrac{1}{2}\theta = \sin(n\pi \pm \tfrac{1}{2}\alpha), \quad \cos\tfrac{1}{2}\theta = \cos(n\pi \pm \tfrac{1}{2}\alpha).$$

These formulae give different values, according as n is even or odd. Thus, if n be even,

$$\sin\tfrac{1}{2}\theta = \pm\sin\tfrac{1}{2}\alpha, \quad \cos\tfrac{1}{2}\theta = \cos\tfrac{1}{2}\alpha.$$

If n be odd, $\sin\frac{1}{2}\theta = \mp\sin\frac{1}{2}\alpha$, $\cos\frac{1}{2}\theta = -\cos\frac{1}{2}\alpha$,

showing that $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$ *are each two-valued.*

54. *If the cosine of an angle be given, the tangent of its half is two-valued.*

For
$$\tan^2\tfrac{1}{2}\theta = \frac{1-\cos\theta}{1+\cos\theta}; \quad \therefore \tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, \qquad (130)$$

which proves the proposition.

Or thus:—

$$\tan \tfrac{1}{2}\theta = \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} = \frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \cos^2 \frac{1}{2}\theta} = \frac{\sin \theta}{1 + \cos \theta}. \qquad (131)$$

But if $\cos \theta$ be given, $\sin \theta$ is evidently two-valued. Hence $\tan \frac{1}{2}\theta$ is two-valued.

Or again—

$$\tan \tfrac{1}{2}\theta = \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} = \frac{2 \sin^2 \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} = \frac{1 - \cos \theta}{\sin \theta}, \text{ \&c.} \qquad (132)$$

Lastly, from § 53, we have

$$\tfrac{1}{2}\theta = n\pi \pm \tfrac{1}{2}\alpha.$$

Hence
$$\tan \tfrac{1}{2}\theta = \pm \tan \tfrac{1}{2}\alpha.$$

55. *If the sine of an angle be given, the sine and the cosine of its half are each four-valued.*

Dem.—We have $2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta = \sin \theta$,

$$\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 1;$$

$$\therefore \quad \sin \tfrac{1}{2}\theta + \cos \tfrac{1}{2}\theta = \pm \sqrt{1 + \sin \theta},$$

$$\sin \tfrac{1}{2}\theta - \cos \tfrac{1}{2}\theta = \pm \sqrt{1 - \sin \theta}.$$

Hence
$$2 \sin \tfrac{1}{2}\theta = \pm \sqrt{1 + \sin \theta} \pm \sqrt{1 - \sin \theta}, \qquad (133)$$

$$2 \cos \tfrac{1}{2}\theta = \pm \sqrt{1 + \sin \theta} \mp \sqrt{1 - \sin \theta}. \qquad (134)$$

From these equations it follows that $\sin \frac{1}{2}\theta$, $\cos \frac{1}{2}\theta$ have each four values equal, two by two, in absolute value, but of contrary signs, and that the values of $\sin \frac{1}{2}\theta$ are the same as those of $\cos \frac{1}{2}\theta$, except in sign.

These results may be shown otherwise as follows:—From § 27 we see that all the values θ, having the same sine α, are

$$\theta = 2n\pi + \alpha, \qquad \theta = (2n + 1)\pi - \alpha.$$

Hence
$$\tfrac{1}{2}\theta = n\pi + \tfrac{1}{2}\alpha, \quad \tfrac{1}{2}\theta = n\pi + \tfrac{1}{2}(\pi - \alpha).$$

Whence—

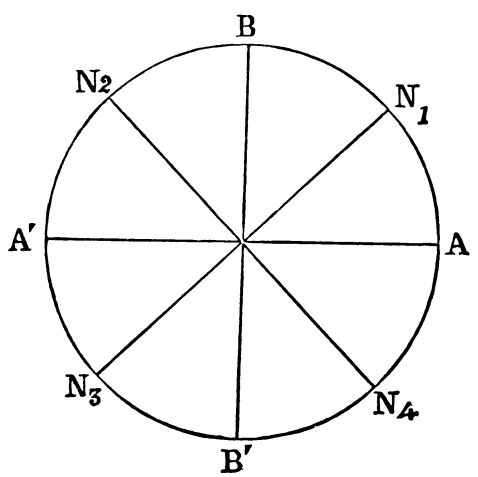
$$\begin{cases} \cos\frac{1}{2}\theta = \cos\left(n\pi + \frac{1}{2}\alpha\right) = \begin{cases} +\cos\frac{1}{2}\alpha, & n \text{ even.} \\ -\cos\frac{1}{2}\alpha, & n \text{ odd.} \end{cases} \\ \cos\frac{1}{2}\theta = \cos\left(n\pi + \frac{1}{2}(\pi - \alpha)\right) = \begin{cases} +\sin\frac{1}{2}\alpha, & n \text{ even.} \\ -\sin\frac{1}{2}\alpha, & n \text{ odd.} \end{cases} \end{cases}$$

$$\begin{cases} \sin\frac{1}{2}\theta = \sin\left(n\pi + \frac{1}{2}\alpha\right) = \begin{cases} +\sin\frac{1}{2}\alpha, & n \text{ even.} \\ -\sin\frac{1}{2}\alpha, & n \text{ odd.} \end{cases} \\ \sin\frac{1}{2}\theta = \sin\left(n\pi + \frac{1}{2}(\pi - \alpha)\right) = \begin{cases} +\cos\frac{1}{2}\alpha, & n \text{ even.} \\ -\cos\frac{1}{2}\alpha, & n \text{ odd.} \end{cases} \end{cases}$$

56. It is necessary to know the proper sign to be given to the radicals in equations (133), (134). This depends on the quadrant of the unit circle in which the arc θ terminates. It suffices to know the signs, which it is necessary to take in

$$\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta = \pm\sqrt{1 + \sin\theta}.$$

$$\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta = \pm\sqrt{1 - \sin\theta}.$$



Let N_1, N_2, N_3, N_4 be the middle points of the four quadrants of the unit circle; M the extremity of the arc $\frac{1}{2}\theta$; R, R' the radicals

$$\sqrt{1 + \sin\theta}, \quad \sqrt{1 - \sin\theta}.$$

1°. If M falls on the arc $N_1 N_4$, $\frac{1}{2}\theta$ is comprised between $2n\pi + \pi/4$ and $2n\pi - \pi/4$; then $\sin\frac{1}{2}\theta$ is in absolute value less than $\cos\frac{1}{2}\theta$, which is +. Hence we take $+R$, $-R'$.

2°. If M falls on the arc $N_1 B N_2$, $\frac{1}{2}\theta$ lies between $2n\pi + \pi/4$ and $2n\pi + 3\pi/4$, and in absolute value $\sin\frac{1}{2}\theta$ is greater than $\cos\frac{1}{2}\theta$, and we take $+R$, $+R'$.

3°. If M falls upon the arc $N_2 A' N_3$, $\frac{1}{2}\theta$ lies between $(2n+1)\pi - \pi/4$ and $(2n+1)\pi + \pi/4$; and evidently we must take $-R$, $+R'$.

4°. If M falls on the arc $N_3 B' N_4$, $\frac{1}{2}\theta$ is comprised between $(2n\pi+1)\pi + \pi/4$ and $(2n+1)\pi + 3\pi/4$; and the radicals are $-R$, $-R'$.

These results may be proved analytically as follows:—

If $\frac{1}{2}\theta$ lies between $2n\pi + \pi/4$ and $(2n+1)\pi + \pi/4$, $\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta$ is positive; and if between $(2n-1)\pi + \pi/4$ and $2n\pi + \pi/4$, $\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta$ is negative.

For $\sin\left(\frac{1}{2}\theta - \frac{\pi}{4}\right) = (\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta)\frac{1}{\sqrt{2}}$. Now if $\frac{1}{2}\theta$ lies between $2n\pi + \pi/4$ and $(2n+1)\pi + \pi/4$, $\frac{1}{2}\theta - \pi/4$ lies between $2n\pi$ and $(2n+1)\pi$, and its sine is +; $\therefore$ $\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta$ is +. In the same manner, if $\frac{1}{2}\theta$ lies between $(2n-1)\pi + \pi/4$ and $2n\pi + \pi/4$, $\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta$ is −.

If $\frac{1}{2}\theta$ lies between $2n\pi - \pi/4$ and $2n\pi + 3\pi/4$, $\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta$ *is +; and if between $2n\pi - 5\pi/4$ and $2n\pi - \pi/4$, $\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta$ is −.*

This may be proved by the equation—

$$\sin\left(\frac{1}{2}\theta + \frac{\pi}{4}\right) = \left(\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta\right)\frac{1}{\sqrt{2}}.$$

57. *If the tangent of an angle be given, the tangent of its half is two-valued.*

DEM.—We have $\tan\theta = \dfrac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$. Equation (101).

Hence, putting $\tan\frac{1}{2}\theta = x$ and $\tan\theta = a$, we have

$$x^2 + \frac{2}{a}x - 1 = 0, \qquad (135)$$

a quadratic which has two real roots, and which proves the proposition.

Or thus:—Let α be a determinate angle, whose tangent is a; then we have (§ 29)—

$$\theta = n\pi + \alpha, \quad \text{or} \quad \begin{cases} \theta = 2p\pi + \alpha, \\ \theta = (2p+1)\pi + \alpha; \end{cases}$$

whence

$$\tan\tfrac{1}{2}\theta = \begin{cases} \tan\left(p\pi + \dfrac{\alpha}{2}\right) = \tan\tfrac{1}{2}\alpha, \\ \tan\left\{p\pi + \left(\dfrac{\pi}{2} + \dfrac{\alpha}{2}\right)\right\} = -\cot\tfrac{1}{2}\alpha. \end{cases}$$

Hence we see also that the product of the two values of $\tan\frac{1}{2}\theta = -1$. If the value of θ be given, $\tan\frac{1}{2}\theta$ will have but one value, which will be + if $\frac{1}{2}\theta$ terminate in the 1st or 3rd quadrant, and – if in the second or fourth.

58. If the cosine of an angle be given, the cosine of its one-third is three-valued.

DEM.—In the formula $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$, put $\alpha = \dfrac{\theta}{3}$, and we get $\cos\theta = 4\cos^3\dfrac{\theta}{3} - 3\cos\dfrac{\theta}{3}$. Hence, putting $\cos\theta = a$ and $\cos\dfrac{\theta}{3} = x$, we get

$$x^3 - \frac{3}{4}x - \frac{a}{4} = 0. \qquad (136)$$

This equation has three real roots, all of which are admissible.

For if we put $\cos\theta = a$, and if α be a determinate angle whose cosine is a, we have (§ 28) $\theta = 2n\pi \pm \alpha$.

Hence $$\cos\frac{\theta}{3} = x = \cos\left(\frac{2n\pi}{3} \pm \frac{\alpha}{3}\right); \tag{137}$$

and putting for n the values θ, 1, 2, we get for x the values

$$\cos\frac{\alpha}{3}, \quad \cos\frac{2\pi + \alpha}{3}, \quad \cos\frac{4\pi + \alpha}{3}, \tag{138}$$

which are all real, and are the three roots of the cubic (136).

If we put for n higher values, 3, 4, 5, &c., we get evidently the values (138) over again.

Cor.—The three factors of the equation (136) are

$$x - \cos\frac{\alpha}{3}, \quad x - \cos\frac{2\pi + \alpha}{3}, \quad x - \cos\frac{4\pi + \alpha}{3}.$$

Hence, from the theory of equations, we infer—

$$1^\circ. \quad \cos\frac{\alpha}{3} + \cos\frac{2\pi + \alpha}{3} + \cos\frac{4\pi + \alpha}{3} = 0. \tag{139}$$

$$2^\circ. \quad \cos\frac{\alpha}{3} . \frac{\cos 2\pi + \alpha}{3} + \cos\frac{2\pi + \alpha}{3} . \cos\frac{4\pi + \alpha}{3}$$
$$+ \cos\frac{4\pi + \alpha}{3} . \cos\frac{\alpha}{3} = -\frac{3}{4}. \tag{140}$$

$$3^\circ. \quad 4\cos\frac{\alpha}{3} . \cos\frac{2\pi + \alpha}{3} . \cos\frac{4\pi + \alpha}{3} = \cos\alpha. \tag{141}$$

From this article it follows that the problem of trisecting an angle depends on the solution of a cubic equation, and conversely, that the solution of a cubic equation can be effected by means of tables of the circular functions.

EXERCISES.—VIII.

1. If $\cos\alpha = \frac{1}{2}$, find the values of $\cos\frac{1}{2}\alpha$ and $\sin\frac{1}{2}\alpha$.

2. „ $2\cos\alpha = -\sqrt{1+\sin 2\alpha} - \sqrt{1-\sin 2\alpha}$. In what quadrant does α terminate?

3. „ $\tan 2\theta = \frac{24}{7}$, find the values of $\tan\theta$, $\sin\theta$, $\cos\theta$.

4. „ $\sin\alpha = \frac{1}{2}$, prove $\tan\frac{1}{2}\alpha = 2 \pm \sqrt{3}$.

5. Prove $\operatorname{cosec}\alpha + \cot\alpha = \cot\frac{1}{2}\alpha$. (142)

6. „ $\operatorname{cosec}\alpha - \cot\alpha = \tan\frac{1}{2}\alpha$. (143)

7. „ $\sec\alpha + \tan\alpha = \tan(45° + \frac{1}{2}\alpha)$. (144)

8. „ $\sec\alpha - \tan\alpha = \tan(45° - \frac{1}{2}\alpha)$. (145)

9. „ $\tan^2\frac{1}{2}\theta = \tan^2\frac{1}{2}\alpha \cdot \tan^2\frac{1}{2}\beta$, if $\cos\theta = \dfrac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}$.

10. „ $\tan^2\frac{1}{2}\theta = \tan^2\frac{1}{2}\alpha \cdot \tan^2\frac{1}{2}\beta$, if $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$.

11. „ $\cot\frac{1}{2}\alpha - \tan\frac{1}{2}\alpha = 2\cot\alpha$.

12. „ $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha = \cot\alpha - 16\cot 16\alpha$.

13. Sum the series $\operatorname{cosec}\alpha + \operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha$, &c., to n terms. [Make use of (142).]

14–19. Solve the following equations :—

14. $\tan\left(\theta + \dfrac{\pi}{4}\right) = 1 + \sin 2\theta$.

15. $\tan\theta = (2 + \sqrt{3})\tan\dfrac{\theta}{3}$.

16. $\cos\frac{1}{2}\theta = a(1 - \cos\theta)$.

17. $\tan\frac{1}{2}\theta = a(1 - \cos\theta)$.

18. $\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta = a(1 + \sin\theta)$.

19. $\cos\theta(1 - \tan\frac{1}{2}\theta) = a(1 + \tan\frac{1}{2}\theta)$.

20. Calculate $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$ in functions of $\tan\theta$.

21. „ $\sin\dfrac{\theta}{4}$, $\cos\dfrac{\theta}{4}$ „ $\cos\theta$.

22. „ $\sin\theta$, $\cos\theta$ „ $\cos\dfrac{2\theta}{3}$.

23–30. Prove the following series of values:—

23. $2\cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}$, $\quad 2\sin\frac{\pi}{8} = \sqrt{2-\sqrt{2}}$. (146)

24. $2\cos\frac{\pi}{16} = \sqrt{2+\sqrt{2+\sqrt{2}}}$, $\quad 2\sin\frac{\pi}{16} = \sqrt{2-\sqrt{2+\sqrt{2}}}$. (147)

25. $2\cos\frac{\pi}{32} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$, $\quad 2\sin\frac{\pi}{32} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$, &c. (148)

26. $2\cos\frac{\pi}{12} = \sqrt{2+\sqrt{3}}$, $\quad 2\sin\frac{\pi}{12} = \sqrt{2-\sqrt{3}}$. (149)

27. $2\cos\frac{\pi}{24} = \sqrt{2+\sqrt{2+\sqrt{3}}}$, $\quad 2\sin\frac{\pi}{24} = \sqrt{2-\sqrt{2+\sqrt{3}}}$. (150)

28. $2\cos\frac{\pi}{48} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$, $\quad 2\sin\frac{\pi}{48} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$, &c. (151)

29. $2\cos\frac{\pi}{20} = \sqrt{2+\sqrt{\tfrac{1}{2}(5+\sqrt{5})}}$, $\quad 2\sin\frac{\pi}{20} = \sqrt{2-\sqrt{\tfrac{1}{2}(5+\sqrt{5})}}$. (152)

30. $2\cos\frac{\pi}{40} = \sqrt{2+\sqrt{2+\sqrt{\tfrac{1}{2}(5+\sqrt{5})}}}$, $\quad 2\sin\frac{\pi}{40} = \sqrt{2-\sqrt{2+\sqrt{\tfrac{1}{2}(5+\sqrt{5})}}}$. (153)

31. Prove $\tan\frac{\pi}{8} = \sqrt{2} - 1$. (154)

32. ,, $\tan^2\frac{\pi}{16} = 7 + 4\sqrt{2} - (2+\sqrt{2})^{\frac{3}{2}}$.

33. ,, $\cos^2\frac{\pi}{10}\sin^2\frac{\pi}{5} - \cos\frac{\pi}{5}\sin\frac{\pi}{10} = \frac{1}{16}$.

34. If $\sin\alpha = a$, prove that the three values of $\sin\alpha/3$ are the roots of the cubic

$$4x^3 - 3x + a = 0. \qquad (155)$$

35. If $\tan\alpha = a$, prove that the three values of $\tan\alpha/3$ are the roots of the cubic

$$x^3 - 3ax^2 - 3x + a = 0. \qquad (156)$$

36. If

$$\sec(\theta+\alpha) + \sec(\theta-\alpha) = 2\cos\theta;$$

prove that

$$\cos\theta = \sqrt{2}\cos\tfrac{1}{2}\alpha.$$

Solve the following equations :—

37. $\tan(45° - \theta) + \tan(45° + \theta) = 4$.

38. $\cos 3\theta + 8 \cos^3\theta = 0$.

39. $\cos\frac{1}{2}(n+1)\theta + \cos\frac{1}{2}(n-1)\theta = \sin\theta$.

40. $2 \operatorname{cosec}\theta - \cot\theta = \sqrt{3}$.

41. $\tan\theta + \tan 2\theta + \tan 3\theta + \tan 4\theta = 0$.

42. $8 \sin\left(\theta + \frac{\pi}{3}\right)\cos^3\theta + 8\cos\left(\theta + \frac{\pi}{3}\right)\sin^3\theta + 6\sin\left(2\theta + \frac{\pi}{3}\right) = \sqrt{3}$.

SECTION V.—FORMULAE FOR THE TRANSFORMATION OF PRODUCTS.

59. From the fundamental formulae (62)–(65), others can be inferred by the simple processes of addition and subtraction.

Thus, taking the sum and the difference of (62) and (63), and also of (64) and (65), we get

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin\alpha \cos\beta. \qquad (157)$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos\alpha \sin\beta. \qquad (158)$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos\alpha \cos\beta. \qquad (159)$$

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin\alpha \sin\beta. \qquad (160)$$

60. Applications—1°. The formulae (157)–(160) are used in proving identities by transforming products into terms of first degree.

I.—Thus, to transform $4\cos\alpha\cos\beta\cos\gamma$ into a sum, we have

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta).$$

Hence (Equation 159.)

$$4\cos\alpha\cos\beta\cos\gamma = 2\cos(\alpha+\beta)\cos\gamma + 2\cos(\alpha-\beta)\cos\gamma$$

$$=\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)+\cos(\beta+\gamma-\alpha). \qquad (161)$$

II.—Prove that

$$\sin a \,.\, \sin(b-c) + \sin b \,.\, \sin(c-a) + \sin c \,.\, \sin(a-b) = 0. \qquad (162)$$

We have

$$2 \sin a \sin (b - c) = \cos (a - b + c) - \cos (a + b - c),$$

$$2 \sin b \sin (c - a) = \cos (b - c + a) - \cos (b + c - a),$$

$$2 \sin c \sin (a - b) = \cos (c - a + b) - \cos (c + a - b);$$

and, when added, the terms on the right cancel each other.

2°. The formulae (157)-(160) are often employed in solving equations.

EXAMPLE.

Solve the equation $\sin (x + 30°) \cos (x - 45°) = \frac{2}{3}$.

We have, making $\alpha = x + 30°$, and $\beta = x - 45°$, in (157),

$$\sin (2x - 15°) + \sin 75° = \tfrac{4}{3};$$

$$\therefore \quad \sin (2x - 15°) = \tfrac{4}{3} - \sin 75°.$$

The trigonometrical Tables will give for $2x - 15°$ a value, say α, comprised between 0 and $\pi/2$; then

$$2x - 15° = 2n\pi + \alpha, \quad \text{or} \quad (2n + 1)\pi - \alpha;$$

$$\therefore \quad x = n\pi + \frac{\alpha + 15°}{2}, \quad \text{or} \quad n\pi + \frac{\pi - \alpha + 15°}{2}.$$

Exercises.—IX.

1. Transform $4 \cos \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c$ into a sum.

2. ,, $4 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c$,,

3 ,, $4 \sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c$,,

4–8. Prove the following identities:—

4. $\cos a \sin (b - c) + \cos b \sin (c - a) + \cos c \sin (a - b) = 0.$

5. $\sin (a - b) \sin (c - d) = \cos (a - c) \cos (b - d) - \cos (a - d) \cos (b - c).$

6. $\sin a \sin b + \sin c \sin (a + b + c) = \sin (a + c) \sin (b + c).$

7. $\cos (a + b) \cos (a - b) + \cos (b + c) \cos (b - c) + \cos (c + a) \cos (c - a)$
$= \cos 2a + \cos 2b + \cos 2c.$

8. $\sin(a+b)\cos(a-b)+\sin(b+c)\cos(b-c)+\sin(c+a)\cos(c-a)$
$$= \sin 2a + \sin 2b + \sin 2c.$$

9–10. Solve the equations:—

9. $$\sin(x+8^\circ)\cos(x-8^\circ) = \cos 45^\circ \sin 61^\circ.$$

10. $$\cos(x+15^\circ)\cos(x-15^\circ) = \sin 45^\circ \sin 75^\circ.$$

SECTION VI.—FORMULAE FOR THE TRANSFORMATION OF SUMS INTO PRODUCTS.

61. By an easy transformation the formulae (157)–(160) give four others for the sum and the difference of the sines and the cosines of two angles.

Thus, let $$(\alpha+\beta)=\sigma, \quad (\alpha-\beta)=\delta;$$

then $$\alpha = \tfrac{1}{2}(\sigma+\delta), \quad \beta = \tfrac{1}{2}(\sigma-\delta).$$

And, substituting these values in (157)–(160), we get, after putting for uniformity of notation, α, β for σ, δ,

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta). \qquad (163)$$

$$\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta). \qquad (164)$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta). \qquad (165)$$

$$\cos\beta - \cos\alpha = 2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta). \qquad (166)$$

These formulae are so important, and recur so frequently, that it will be advisable for the student to commit the following enunciations to memory:—

Of any two angles, the

Sum of the sines = 2 *sin* ½ sum . *cos* ½ diff.

Diff. ,, = 2 *cos* ½ sum . *sin* ½ diff.

Sum of the cosines = 2 *cos* ½ sum . *cos* ½ diff.

Diff. ,, = 2 *sin* ½ sum . *sin* ½ diff.

62. If we take the quotient of each pair of the formulae (163)–(166), we get the following results:—The first is reduced by dividing both numerator and denominator by

$$2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta),$$

and the reduction of the others is obvious.

$$(\sin \alpha + \sin \beta)/(\sin \alpha - \sin \beta) = \tan \tfrac{1}{2}(\alpha + \beta)/\tan \tfrac{1}{2}(\alpha - \beta). \quad (167)$$

$$(\sin \alpha + \sin \beta)/(\cos \alpha + \cos \beta) = \tan \tfrac{1}{2}(\alpha + \beta). \quad (168)$$

$$(\sin \alpha + \sin \beta)/(\cos \beta - \cos \alpha) = \cot \tfrac{1}{2}(\alpha - \beta). \quad (169)$$

$$(\sin \alpha - \sin \beta)/(\cos \alpha + \cos \beta) = \tan \tfrac{1}{2}(\alpha - \beta). \quad (170)$$

$$(\sin \alpha - \sin \beta)/(\cos \beta - \cos \alpha) = \tan \tfrac{1}{2}(\alpha + \beta). \quad (171)$$

$$(\cos \alpha + \cos \beta)/(\cos \beta - \cos \alpha) = \cot \tfrac{1}{2}(\alpha + \beta) \cot \tfrac{1}{2}(\alpha - \beta). \quad (172)$$

The formula (167) is of frequent use. *The sum of the sines of two arcs is to their difference as tan ½ sum of the arcs is to tan ½ difference.*

63. If in the formulae (163), (165), (166) we make $\beta = 0$, we get, remembering that $\cos(0) = 1$, and $\sin(0) = 0$,

$$\sin \alpha = 2 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha;$$

$$1 + \cos \alpha = 2 \cos^2 \tfrac{1}{2}\alpha, \quad \text{or} \quad \cos \alpha = 2 \cos^2 \tfrac{1}{2}\alpha - 1;$$

$$1 - \cos \alpha = 2 \sin^2 \tfrac{1}{2}\alpha, \quad \text{or} \quad \cos \alpha = 1 - 2 \sin^2 \tfrac{1}{2}\alpha.$$

From the two last we get, by subtraction,

$$\cos \alpha = \cos^2 \tfrac{1}{2}\alpha - \sin^2 \tfrac{1}{2}\alpha.$$

These formulae have been obtained differently in § 49, equations (99), (103), (104), (100). They are so important that we recommend the student to commit the following enunciations to memory:—

$$\text{sine angle} = 2 \sin \tfrac{1}{2} \text{ angle} \,.\, \cos \tfrac{1}{2} \text{ angle}. \quad (99)$$

$$1 + \cos.\text{angle} = 2 \cos^2 \tfrac{1}{2} \text{ angle}. \quad (103)$$

$$1 - \cos.\text{angle} = 2\sin^2\tfrac{1}{2}\text{angle}. \qquad (104)$$

$$\cos\text{angle} = \cos^2\tfrac{1}{2}\text{angle} - \sin^2\tfrac{1}{2}\text{angle}. \qquad (100)$$

From (103), (104) we get, by changing a into $(\frac{1}{2}\pi - a)$,

$$1 + \sin a = 2\cos^2\tfrac{1}{2}(\tfrac{1}{2}\pi - a). \qquad (173)$$

$$1 - \sin a = 2\sin^2\tfrac{1}{2}(\tfrac{1}{2}\pi - a). \qquad (174)$$

64. Applications.—The formulae of this section are of the highest importance. They are frequently employed in transforming polynomials into monomials, in order to render them adapted to logarithmic calculation. They are also used in the resolution of equations.

EXAMPLE 1.—Reduce

$\sin A + \sin B + \sin C - \sin(A + B + C)$ to a monomial.

We have

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A - B),$$

$$\sin(A + B + C) - \sin C = 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A + B + 2C).$$

Hence, by subtraction,

$$\sin A + \sin B + \sin C - \sin(A + B + C)$$

$$= 2\sin\tfrac{1}{2}(A + B)\{\cos\tfrac{1}{2}(A - B) - \cos\tfrac{1}{2}(A + B + 2C)\}$$

$$= 4\sin\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}(B + C)\sin\tfrac{1}{2}(C + A). \qquad (175)$$

This formula may be regarded as fundamental. By changing the notation, or by particular hypothesis, for A, B, C, we may infer from it several other formulae. Thus:—

1°. Replacing A, B, C by $\frac{\pi}{2} - A$, $\frac{\pi}{2} - B$, $\frac{\pi}{2} - C$,

we get $\quad \cos A + \cos B + \cos C + \cos(A + B + C)$

$$= 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}.\cos\frac{C+A}{2}. \qquad (176)$$

2°. If $(A + B + C) = \pi$, or if A, B, C be the angles of a triangle, we get

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}, \tag{177}$$

and $$\cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \tag{178}$$

Also $$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C, \tag{179}$$

and $$\cos 2A + \cos 2B + \cos 2C + 1 + 4 \cos A \cos B \cos C = 0. \tag{180}$$

3°. The formula is applicable if

$$A + B + C = 0.$$

We can then replace

$$A, B, C \text{ by } \beta - \gamma, \quad \gamma - \alpha, \quad \alpha - \beta,$$

which gives

$$\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)$$
$$+ 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} . \sin \frac{\alpha - \beta}{2} = 0. \tag{181}$$

$$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) + 1$$
$$= 4 \cos \frac{\beta - \gamma}{2} \cos \frac{\gamma - \alpha}{2} . \cos \frac{\alpha - \beta}{2}. \tag{182}$$

4°. If A, B, C be half the angles of a triangle, we have, putting $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ for A, B, C,

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}. \tag{183}$$

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1 = 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}. \tag{184}$$

Example 2.—Reduce to a monomial

$$1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C.$$

Adding and subtracting $\cos^2 A \cos^2 B$, we get

$$(1 - \cos^2 A)(1 - \cos^2 B) - (\cos C - \cos A \cos B)^2$$
$$= \sin^2 A \sin^2 B - (\cos C - \cos A \cos B)^2$$

$$= (\sin A \sin B + \cos C - \cos A \cos B)(\sin A \sin B - \cos C + \cos A \cos B)$$

$$= \{\cos C - \cos (A + B)\} \{\cos (A - B) - \cos C\}$$

$$= 4 \sin \frac{A+B+C}{2} \sin \frac{A+B-C}{2} . \sin \frac{A-B+C}{2} . \sin \frac{C-A+B}{2}. \tag{185}$$

Similarly, $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1$

$$= 4 \cos \frac{A+B+C}{2} \cos \frac{A+B-C}{2} . \cos \frac{A-B+C}{2} \cos \frac{C-A+B}{2}. \tag{186}$$

Example 3.—Solve the equation

$$\sin x + \sin 2x + \sin 3x = 1 + \cos 2x.$$

We have $\quad \sin x + \sin 3x = 2 \sin 2x \cos x,$

and $\quad \sin 2x = 2 \sin x \cos x.$

Hence $\quad 2(\sin 2x + \sin x) \cos x = 1 + \cos 2x = 2 \cos^2 x.$

Hence we get as a first solution

$$\cos x = 0, \quad \text{or} \quad x = (2n + 1) \frac{\pi}{2},$$

and there remains

$$\sin 2x + \sin x = \cos x.$$

Hence $\quad \sin^2 2x = 1 - \sin 2x;$

therefore $\quad \sin 2x = - \tfrac{1}{2} \pm \frac{\sqrt{5}}{2}.$

Exercises.—X.

1. Prove $\sin 4\alpha + \sin 2\alpha = 2 \sin 3\alpha \cos \alpha$.

2. ,, $\cos 2\alpha - \cos 4\alpha = 2 \sin 3\alpha \sin \alpha$.

3. ,, $\sin 3\alpha + \sin 5\alpha = 8 \sin \alpha \cos^2 \alpha \cos 2\alpha$.

4. ,, $\cos \alpha + \cos 3\alpha + \cos 5\alpha + \cos 7\alpha = 4 \cos \alpha . \cos 2\alpha . \cos 4\alpha$.

5. Prove $\sin\alpha + \cos\beta = 2\cos\left\{\frac{\pi}{4} - \frac{1}{2}\left(\alpha+\beta\right)\right\}\cos\left\{\frac{\pi}{4} - \frac{1}{2}\left(\alpha-\beta\right)\right\}$. (187)

6 ,, $\sin\alpha - \cos\beta = 2\sin\left\{\frac{\pi}{4} - \frac{1}{2}\left(\alpha-\beta\right)\right\}\sin\left\{\frac{1}{2}\left(\alpha+\beta\right) - \frac{\pi}{4}\right\}$. (188)

7. ,, $\sin\alpha + \cos\beta = 2\sin\left\{\frac{\pi}{4} + \frac{1}{2}\left(\alpha+\beta\right)\right\}\sin\left\{\pi + \frac{1}{2}\left(\alpha-\beta\right)\right\}$. (189)

8. ,, $\sin\alpha - \cos\beta = 2\cos\left\{\frac{\pi}{4} + \frac{1}{2}\left(\alpha-\beta\right)\right\}\cos\left\{\frac{3\pi}{4} - \frac{1}{2}\left(\alpha+\beta\right)\right\}$. (190)

9. ,, $\cos 55^\circ . \cos 65^\circ + \cos 65^\circ . \cos 175^\circ + \cos 55^\circ . \cos 175^\circ = -\frac{3}{4}$.

10. ,, $\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}/\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\} = \tan\alpha/\tan\beta$.

11. ,, $\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}/\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\} = \cot\alpha . \cot\beta$.

12–15. Reduce the following to monomials:—

12. $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha$.

13. $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha + \sin 5\alpha$.

14. $1 + \cos\alpha + \cos\beta + \cos\gamma \pm 4\cos\frac{\alpha}{2} . \cos\frac{\beta}{2} . \cos\frac{\gamma}{2}$.

15. $1 - \cos\alpha - \cos\beta - \cos\gamma \pm 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2} . \sin\frac{\gamma}{2}$.

16. Prove $\cos(3n+1)\alpha + \cos(3n-1)\alpha + 3\cos(n+1)\alpha + 3\cos(n-1)\alpha$
$= 8\cos\alpha\cos^3 n\alpha$.

17. ,, $\cos^2\alpha + \sin^2 3\alpha - \cos^2 2\alpha = 2\cos\alpha\sin 2\alpha . \sin 3\alpha$.

18–23. Solve the following equations:—

18. $(\sin x + \sin 2x)/\cos\frac{1}{2}x = a$.

19. $\sin\frac{x}{2} = a(\cos 2x - \cos x)$.

20. $\sin(3x+\alpha) + \sin(3x-\alpha) + \sin(\alpha - x) - \sin(\alpha + x) = \cos\alpha$.

21. $\sin x + \sin 2x = a + \cos x$.

22. $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$.

23. $1 + 2\sin x = 2\sin 3x$.

24. Prove $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$.

25. If $\tan A$, $\tan B$, $\tan C$ be in A, P; prove that

$$\sin(B + C - A),\ \sin(C + A - B),\ \sin(A + B - C),$$

are in AP.

26. If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$, prove that each is equal to $\cos A \cos B \cos C$.

27. In the same case, prove

$$\sec^2 A + \sec^2 B + \sec^2 C = 2 \sec A \sec B \sec C.$$

28. If $\cot A$, $\cot B$, $\cot C$ be in AP; prove that

$$\sin(B + C) \operatorname{cosec} A,\ \sin(C + A) \operatorname{cosec} B,\ \sin(A + B) \operatorname{cosec} C,$$

are in AP.

29. If $\cos\theta = \dfrac{\cos u - e}{1 - e\cos u}$, prove that $\tan\frac{1}{2}\theta = \sqrt{\dfrac{1+e}{1-e}} \cdot \tan\frac{1}{2}u$.

30. If $\alpha = \dfrac{2\pi}{15}$, prove that $\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha = \frac{1}{2}$.

31–34. Having given $\sin\alpha + \sin\beta = a$, $\cos\alpha + \cos\beta = b$;

31. prove that $\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2} = 4a/(a^2 + b^2 + 2b)$.

32. ,, $\tan\alpha + \tan\beta = 8ab/((a^2 + b^2)^2 - 4a^2)$.

33. ,, $\cos 2\alpha + \cos 2\beta = (b - a^2)\left(1 - \dfrac{2}{a^2 + b^2}\right)$.

34. ,, $\cos 3\alpha + \cos 3\beta = b^3 - 3a^2 b - 3b + 12a^2 b/(a^2 + b^2)$.

35. Prove that

$$\left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 - \tan^2\frac{\alpha}{2^2}\right)\left(1 - \tan^2\frac{\alpha}{2^3}\right)\ldots \text{to inf.} = \alpha\cot\alpha.$$

36. Prove that $\cos\dfrac{x}{2} \cdot \cos\dfrac{x}{2^2} \cdot \cos\dfrac{x}{2^3} \ldots \text{to inf.} = \dfrac{\sin x}{x}$. (191)

37. Prove Euler's formula—

$$\sin A + \sin(36^\circ - A) + \sin(72^\circ + A) = \sin(36^\circ + A) + \sin(72^\circ - A). \quad (192)$$

38. Prove Legendre's formula—

$$\cos A = \cos(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A). \quad (193)$$

39. Prove Legendre's formula—

$$\cos A = \cos(36^\circ + A) + \cos(36^\circ - A) - \cos(72^\circ - A) - \cos(72^\circ + A). \quad (194)$$

40. If $\alpha = \frac{\pi}{13}$, prove that $\cos\alpha + \cos 3\alpha + \cos 9\alpha = \frac{1+\sqrt{13}}{4}$, (195)

and $$\cos 5\alpha + \cos 7\alpha + \cos 11\alpha = \frac{1-\sqrt{13}}{4}. \qquad (196)$$

This question is proved by showing that the sum and the product of

$$\cos\alpha + \cos 3\alpha + \cos 9\alpha \quad \text{and} \quad \cos 5\alpha + \cos 7\alpha + \cos 11\alpha$$

are $\frac{1}{2}$ and $-\frac{3}{4}$, respectively.

41. Prove that the inscription of a polygon of 13 sides in a circle depends on the solution of a biquadratic equation.

If $\alpha = \frac{\pi}{13}$, 9α and 4α are supplements; $\therefore\ \cos 9\alpha = -\cos 4\alpha$.

Hence, from (195), $\cos\alpha + \cos 3\alpha - \cos 4\alpha = \frac{1+\sqrt{13}}{4}$,

or $$4\cos^3\alpha - 2\cos\alpha - 1 + 8\cos^2\alpha - 8\cos^4\alpha = \frac{1+\sqrt{13}}{4};$$

and putting $2\cos\alpha = x$, we get

$$x^4 - x^3 - 4x^2 + 2x + \frac{5+\sqrt{13}}{2} = 0. \qquad (197)$$

Section VII.—Formulae between Inverse Functions.

65. From the definitions of inverse functions (§ 30), we see that every relation between the direct circular functions has a corresponding relation for the inverse. Thus, from the equation—

$$\tan(\alpha_1 + \alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \tan\alpha} \quad (66),$$

we get $$\alpha_1 + \alpha_2 = \tan^{-1}\left\{\frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1\tan\alpha_2}\right\}.$$

Now, put $\tan\alpha_1 = a_1, \quad \tan\alpha_2 = a_2,$

then $$\alpha_1 = \tan^{-1}a_1, \quad \alpha_2 = \tan^{-1}a_2.$$

Hence $$\tan^{-1}a_1 + \tan^{-1}a_2 = \tan^{-1}\left\{\frac{a_1 + a_2}{1 - a_1 a_2}\right\}, \qquad (198)$$

which is the inverse formula corresponding to (66) for the tangent of the sum of two angles.

In the same manner,

$$\tan^{-1} a_1 - \tan^{-1} a_2 = \tan^{-1} \frac{a_1 - a_2}{1 + a_1 a_2}. \qquad (199)$$

66. The equation (198) can be stated differently. Thus, if a_1, a_2 be the roots of the quadratic

$$x^2 - p_1 x + p_2 = 0,$$

then

$$\tan^{-1} a_1 + \tan^{-1} a_2 = \tan^{-1} \frac{p_1}{1 - p_2}. \qquad (200)$$

We can generalize this result as follows :—

To both sides of (200) add $\tan^{-1} a_3$, and we get

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \tan^{-1} a_3 = \tan^{-1} a_3 + \tan^{-1} \frac{p_1}{1 - p_2}$$

$$= \tan^{-1} \left\{ a_3 + \frac{p_1}{1 - p_2} \right\} \Big/ \left\{ 1 - \frac{a_3 p_1}{1 - p_2} \right\} = \tan^{-1} \left\{ \frac{a_1 + a_2 + a_3 - a_1 a_2 a_3}{1 - a_1 a_2 - a_2 a_3 - a_3 a_1} \right\}.$$

Hence, if a_1, a_2, a_3 be the roots of the cubic

$$x^3 - p_1 x^2 + p_2 x - p_3 = 0,$$

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \tan^{-1} a_3 = \tan^{-1} \left\{ \frac{p_1 - p_3}{1 - p_2} \right\}. \qquad (201)$$

Similarly, if a_1, a_2, a_3, a_4 be the roots of the quartic

$$x^4 - p_1 x^3 + p_2 x^2 - p_3 x + p_4 = 0,$$

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \tan^{-1} a_3 + \tan^{-1} a_4 = \tan^{-1} \left\{ \frac{p_1 - p_3}{1 - p_2 + p_4} \right\}. \qquad (202)$$

And so on for the sum of any number of inverse tangents.

Exercises.—XI.

1. Prove $2\tan^{-1}a = \tan^{-1}\left(\dfrac{2a}{1-a^2}\right)$. (203)

2. ,, $3\tan^{-1}a = \tan^{-1}\left(\dfrac{3a-a^3}{1-3a^2}\right)$. (204)

3. ,, $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$. (205)

4. ,, $\tan^{-1}x + \tan^{-1}\dfrac{1}{x} = \dfrac{\pi}{2}$.

5. ,, $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$. (206)

6. ,, $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{70} + \tan^{-1}\dfrac{1}{99} = \dfrac{\pi}{4}$. (207)

7. ,, $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}$. (208)

8. ,, $\tan^{-1}\dfrac{1}{1+2} + \tan^{-1}\dfrac{1}{1+6} + \tan^{-1}\dfrac{1}{1+12} + \text{\&c., to inf.} = \dfrac{\pi}{4}$.

9. ,, $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} = \sin^{-1}\dfrac{63}{65}$.

10. ,, $\sin^{-1}\dfrac{1}{\sqrt{82}} + \sin^{-1}\dfrac{4}{\sqrt{41}} = \dfrac{\pi}{4}$.

11. If
$$\tan^2\theta = \tan(\theta-\alpha)\tan(\theta-\beta),$$
prove
$$2\theta = \tan^{-1}\left\{\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\right\}.$$

12. If
$$\sin\theta = \sin\alpha/\sin\gamma, \quad \cos\theta = \sin\beta/\sin\gamma,$$
prove
$$\cos^2\alpha = \sin^2\beta + \cos^2\gamma.$$

13-20.—Find the values of x in the following equations:—

13. $$\cot^{-1}x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n-1).$$

14. $$\sin^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{4}.$$

15. $$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}.$$

16. $$2 \tan^{-1} x = \sin^{-1}\left(\frac{2a}{1 + a^2}\right) + \sin^{-1}\left(\frac{2b}{1 + b^2}\right).$$

17. $$\cot^{-1} \frac{1}{x + 1} + \cot^{-1} \frac{1}{x - 1} = \tan^{-1} 3x - \tan^{-1} x.$$

18. $$\cos^{-1} \frac{1}{a} + \cos^{-1} \frac{a}{x} = \cos^{-1} \frac{1}{b} + \cos^{-1} \frac{b}{x}.$$

19. $$\sin (\pi \cos x) = \cos (\pi \sin x).$$

20. $$\tan^{-1} \frac{x - 1}{x - 2} + \tan^{-1} \frac{x + 1}{x + 2} = \frac{\pi}{4}.$$

SECTION VIII.—TRIGONOMETRIC ELIMINATION.

67. *The process of deriving from n simultaneous equations containing* $(n - 1)$ *quantities, a new equation from which these will be absent, is called elimination.*

Every student is familiar with elementary elimination in algebra; for the method of solving two simultaneous equations, containing two unknown quantities, is to eliminate one of them, and from the resulting equation find the value of the other. Three simultaneous equations are solved by eliminating two of the unknowns, &c. The higher departments of algebraic elimination forms an important branch of analysis.

Trigonometric elimination occurs chiefly in the application of Trigonometry to the higher branches of Mathematics, as for example in Physical Astronomy, in Mechanics, The Theory of Envelopes in Analytic Geometry, &c. As no special rules can be given, we illustrate the process by a few examples.

68. Eliminate ϕ from the equations

$$x = a \cos \phi, \quad y = b \sin \phi.$$

From the given equations we have

$$\frac{x}{a} = \cos \phi, \quad \frac{y}{b} = \sin \phi.$$

Hence, if we square and add, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The simple process employed here is of frequent recurrence.

The following is a more difficult example :—

Eliminate ϕ from the equations

$$y \cos\phi - x \sin\phi = a \cos 2\phi,$$

$$y \sin\phi + x \cos\phi = 2a \sin 2\phi.$$

Solving for x and y, and then adding and subtracting, we get

$$(x + y) = a(\sin\phi + \cos\phi)(1 + \sin 2\phi),$$

$$x - y = a(\sin\phi - \cos\phi)(1 - \sin 2\phi).$$

Hence $(x + y)^2 = a^2(1 + \sin 2\phi)^3, \quad (x - y)^2 = a^2(1 - \sin 2\phi)^3;$

$$\therefore \quad (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

69. As an example of the elimination of two unknown quantities, let us take

$$b + c\cos\alpha = u\cos(\alpha + \theta), \quad b + c\cos\beta = u\cos(\beta + \theta), \quad \alpha - \beta = 2\delta.$$

If we put $\alpha + \beta = 2\sigma,$

we get, by adding the two first equations,

$$2b + 2c\cos\sigma\cos\delta = 2u\cos(\sigma + \theta)\cos\delta;$$

and, by subtracting,

$$2c\sin\sigma\sin\delta = 2u\sin(\sigma + \theta)\sin\delta;$$

$$\therefore \quad c\cos\sigma - u\cos(\sigma + \theta) = -b\sec\delta,$$

and $$c\sin\sigma - u\sin(\sigma + \theta) = 0.$$

Square and add, and we get

$$c^2 + u^2 - 2cu\cos\theta = b^2\sec^2\delta.$$

As another example, eliminate θ and ϕ from the equations

$$x = x' \cos\phi + y' \sin\phi \cos\theta, \quad y = x' \sin\phi - y' \cos\phi \cos\theta,$$

$$z = y' \sin\theta.$$

If we square and add, we get

$$x^2 + y^2 + z^2 = x'^2 + y'^2.$$

In the same manner we may eliminate γ and ϕ from the equations

$$x = a \sin\gamma \cos\phi, \quad y = b \sin\gamma \sin\phi, \quad z = c \cos\gamma;$$

the result is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

70. Eliminate x, y, z from the equation

$$\frac{x^2 - y^2}{a^2} \cos\theta = \frac{z^2}{b^2} \cos\theta',$$

and the proportions

$$x : y : z :: \sin(\theta + \theta') : \sin(\theta - \theta') : \sin 2\theta.$$

From the proportions, we get

$$\frac{x^2 - y^2}{z^2} = \frac{\sin^2(\theta + \theta') - \sin^2(\theta - \theta')}{\sin^2 2\theta} = \frac{\sin 2\theta'}{\sin 2\theta},$$

and from the equation

$$\frac{x^2 - y^2}{z^2} = \frac{a^2 \cos\theta'}{b^2 \cos\theta}.$$

Hence

$$\frac{a^2}{b^2} = \frac{\sin\theta'}{\sin\theta}.$$

EXERCISES.—XII.

Eliminate ϕ from the following pairs of equations:—

1. $x = a\cos^3\phi, \quad y = b\sin^3\phi.$

2. $x = \sec\phi - \cos\phi, \quad y = \operatorname{cosec}\phi - \sin\phi.$

3. $\cot\alpha\cot\phi = m, \quad \sin\alpha\sin\phi = n.$

4. $\tan\phi + \cot\phi = m^3, \quad \sec\phi - \cos\phi = n^3.$

5. $\dfrac{\sec^4\phi - 1}{\sec^4\phi + 1} = \dfrac{x}{a}, \quad \sec^2\phi + \cos^2\phi = \dfrac{2b}{y}.$

6. $x = c\log(\sec\phi + \tan\phi), \quad y = c\sec\phi.$

7. $x = a\tan^2\phi, \quad y = 2a\cot\phi.$

8. $\dfrac{x}{k} = 2\tan\phi + 4\tan^3\phi, \quad \dfrac{y}{k} = \tan^2\phi - 3\tan^4\phi.$

9. $x\sin\phi + y\cos\phi = \sqrt{x^2 + y^2}, \quad \dfrac{\cos^2\phi}{a^2} + \dfrac{\sin^2\phi}{b^2} = \dfrac{1}{x^2 + y^2}.$

10. $\sin\phi\sin\theta = \sin\alpha\sin\beta, \quad \tan\phi\cos\beta = \cot\dfrac{\alpha}{2}.$

11. $\dfrac{\sin(\phi - \alpha)}{\sin(\phi - \beta)} = \dfrac{a}{b}, \quad \dfrac{\cos(\phi - \alpha)}{\cos(\phi - \beta)} = \dfrac{c}{d}.$

12. $\tan\theta = \cos\phi\tan\alpha, \quad \tan\phi\sin\theta = \tan\beta.$

13. $x = 3\cos\phi + \cos 3\phi, \quad y = 3\sin\phi - \sin 3\phi.$

14. $a\sin^2\phi + 2b\sin\phi + c = 0, \quad a'\cos^2\phi + 2b'\cos\phi + c' = 0.$

15. $\dfrac{\sin\theta}{a^2 - 1} = \dfrac{\cos\theta}{2a\sin 2\phi} = \dfrac{1}{1 + 2a\cos 2\phi + a^2}.$

16. $4(\cos\alpha\cos\phi + \cos\theta)(\cos\alpha\sin\phi + \sin\theta)$

$$= 4(\cos\alpha\cos\phi + \cos\psi)(\cos\alpha\sin\phi + \sin\psi)$$
$$= (\cos\theta - \cos\psi)(\sin\theta - \sin\psi).$$

17. $\dfrac{\cos(\alpha - 3\phi)}{\cos^3\phi} = \dfrac{\sin(\alpha - 3\phi)}{\sin^3\phi} = m.$

18. $\dfrac{\cos(\alpha + \phi)}{\sin^3\alpha} = \dfrac{\cos(\beta + \phi)}{\sin^3\beta} = \dfrac{\cos(\gamma + \phi)}{\sin^3\gamma};$

α, β, γ being unequal, and each less than π.

19. $\dfrac{2}{1+x} = \dfrac{\sin\beta \sin\phi}{\cos(\beta-\phi)} = \tan(\phi - \alpha)\tan\beta.$

20. $(a+b)\tan(\phi-\theta) = (a-b)\tan(\phi+\theta), \quad a\cos 2\theta + b\cos 2\phi = c.$

21. $\cos(\theta-\phi+\alpha)\cos(\phi-\alpha) = \cos(\theta-\phi-\alpha)\cos(\phi+\alpha) = c.$

22. $x = \sec\phi + \operatorname{cosec}\phi\tan^3\phi(\sec^2\phi+1), \quad y = \tan\phi - \tan^3\phi(\sec^2\phi+1).$

23. $m\cos\phi = \sqrt{\tfrac{1}{3}(1-m^2)}, \quad \tan^3\dfrac{\phi}{2} = \tan\alpha.$

24. $\dfrac{\sin\phi}{a} + \dfrac{\cos\phi}{b} = \dfrac{2}{\sqrt{ab}}, \quad \dfrac{\cos^2\phi}{a^2} + \dfrac{3\sin 2\phi}{ab} + \dfrac{\sin^2\phi}{b^2} = \dfrac{4}{ab}.$

25. $a\cos\phi + b\sin\phi = c, \quad a'\cos\phi + b'\sin\phi = c'.$

26. Eliminate x from the equation

$$\frac{2a\tan\phi}{\tan\alpha + \tan\beta} = \frac{a^2\tan\phi - x}{\tan\alpha\tan\beta} = a - x.$$

Eliminate θ and ϕ from the following triads of equations:—

27. $\sin\alpha = 2\sin\dfrac{\theta}{2}\cdot\sin\dfrac{\phi}{2}, \quad \cos\alpha = \cos\beta\cos\phi = \cos\gamma\cos\theta.$

28. $x\cos\tfrac{1}{2}(\theta-\phi) = \cos\theta\cos\phi\cos\tfrac{1}{2}(\theta+\phi), \quad y\sin\tfrac{1}{2}(\theta-\phi)$
$= \sin\theta\sin\phi\sin\tfrac{1}{2}(\theta+\phi), \quad a\cos\theta\cos\phi + b\sin\theta\sin\phi = c.$

29. $\dfrac{ax}{\cos\theta} - \dfrac{by}{\cos\theta} = a^2 - b^2 = \dfrac{ax}{\cos\phi} - \dfrac{by}{\sin\phi}, \quad (\theta-\phi) = \dfrac{\pi}{2}.$

30. $\tan\theta\cot\phi = \tan\alpha\cot\alpha', \quad \cos^2\theta = \cos a\sec\beta, \quad \cos^2\phi = \cos\alpha'\sec\beta.$

31. $p\sin^4\theta - q\sin^4\phi = p, \quad p\cos^4\theta - q\cos^4\phi = q, \quad \sin(\theta-\phi) = \tfrac{1}{10}.$

32. $x\cos\theta + y\sin\theta = x\cos\phi + y\sin\phi = 1,$
$4\cos\tfrac{1}{2}(\theta-\phi)\cos\tfrac{1}{2}(\alpha-\theta)\cos\tfrac{1}{2}(\alpha-\phi) = 1.$

33. $x\cos\theta + y\sin\theta = x\cos(\theta-2\phi) - y\sin(\theta-2\phi) = a,$
$a\sin\theta + b\sin(\theta-\phi) = 0.$

34. $\theta + \phi = \alpha, \quad \tan\theta + \tan\phi = a, \quad \cot\theta + \cot\phi = b.$

35. $a\sin^2\theta + a'\cos^2\theta = b, \quad a'\sin^2\phi + a\cos^2\phi = b', \quad a\tan\theta = b\tan\phi.$

36. $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1 = \dfrac{x\cos\phi}{a} + \dfrac{y\sin\phi}{b}, \quad \theta - \phi = 2a.$

37. $\sin\alpha\cos\theta = \sin\beta, \quad \sin\alpha\cos\phi = \sin\gamma, \quad \cos(\theta - \phi) = \sin\beta\sin\gamma.$

38. $x = (a\sin^2\theta + b\cos^2\theta)\cos^2\phi + c\sin^2\phi, \quad y = a\cos^2\theta + b\sin^2\theta,$

$$z = (b - a)\sin\theta\cos\theta\cos\phi.$$

39. $x\cos\theta + y\sin\theta = x\cos\phi + y\sin\phi = 1, \quad a\cos\theta\cos\phi + b\sin\theta\sin\phi$

$$+ h\sin(\theta+\phi) + f(\cos\theta + \cos\phi) + g(\sin\theta + \sin\psi) + c = 0.$$

Section IX.—Trigonometric Identities.

71. The verification of certain identities between the circular functions of one or more arcs is an important process in Trigonometry. It consists in the application of the formulae of sections I.–VII. of this chapter, and of certain analogies which exist between algebraic and trigonometric formulae; for example,

$$\sin(p + q)\sin(p - q) = \sin^2 p - \sin^2 q, \text{ \&c.}$$

Example 1.—Verify

$$\sin^3 a\sin(b - c) + \sin^3 b\sin(c - a) + \sin^3 c\sin(a - b)$$

$$= \sin(a + b + c)\sin(a - b)\sin(b - c)\sin(a - c). \quad (209)$$

Denoting the left-hand side by P, and transforming it by the formula

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

we get

$$P = \tfrac{3}{4}\Sigma\sin a \,.\, \sin(b - c) - \tfrac{1}{4}\Sigma\sin 3a\sin(b - c);$$

but

$$\Sigma\sin a \,.\, \sin(b - c) = 0.$$

Hence

$$P = -\tfrac{1}{8}\Sigma\{\cos(3a - b + c) - \cos(3a + b - c)\}.$$

And, reducing as follows:—

$$\cos(3a - b + c) - \cos(3b + c - a) = 2\sin(a + b + c)\sin(2b - 2a),$$

$$\cos(3b - c + a) - \cos(3c + a - b) = 2\sin(a + b + c)\sin(2c - 2b),$$

$$\cos(3c - a + b) - \cos(3a + b - c) = 2\sin(a + b + c)\sin(2a - 2c).$$

Hence the proposition is evident.

The algebraic formula corresponding to this is

$$a^3(b - c) + b^3(c - a) + c^3(a - b) = (a + b + c)(a - b)(b - c)(a - c).$$

EXAMPLE 2.—Prove

$$\Sigma \sin^2 a \sin(b + c - a) = 2\sin a \sin b \sin c$$
$$+ \sin(b + c - a)\sin(c + a - b)\sin(a + b - c). \quad (210)$$

72. Identities relative to three Angles whose Sum is a Multiple of π.—We have already given several identities between the circular functions of the three angles of a triangle. The following are some others of frequent use:—

1°. Relation between the cosines of three angles whose sum is $n\pi$.

Let $$A + B + C = n\pi;$$

$$\therefore \quad \cos A = \pm\cos(B + C); \quad + \text{ if } n \text{ is even}, \quad - \text{ if } n \text{ is odd};$$

$$\therefore \cos A \mp \cos B\cos C = \mp\sin B\sin C = \sqrt{(1 - \cos^2 B)(1 - \cos^2 C)}; \quad (1)$$

$$\therefore \quad \cos^2 A + \cos^2 B + \cos^2 C \mp 2\cos A\cos B\cos C = 1. \quad (211)$$

The relation (1) gives also

$$\cos A \pm \sin B\sin C = \sqrt{(1 - \sin^2 B)(1 - \sin^2 C)};$$

whence

$$\sin^2 A = \sin^2 B + \sin^2 C \pm 2\sin B\sin C\cos A. \quad (212)$$

This formula may be inferred from (211) by replacing

$$B,\ C \text{ by } B + \frac{\pi}{2}, \quad C + \frac{\pi}{2},$$

which alters by unity the multiples of π, to which the sum of A, B, C is equal.

From the formulae (211), (212) several others may be inferred by processes similar to those in § 64. Thus, replacing

$$A,\ B,\ C \text{ by } \frac{\pi}{2}-A,\quad \frac{\pi}{2}-B,\quad \frac{\pi}{2}-C,$$

we see that when $\quad A + B + C = (4n \pm 1)\dfrac{\pi}{2},$

$$\sin^2 A + \sin^2 B + \sin^2 C \pm 2 \sin A \sin B \sin C = 1. \qquad (213)$$

2°. To find the relation between the tangents of three angles whose sum is $n\pi$.

We have $\quad \tan A = -\tan(B + C) = \dfrac{-(\tan B + \tan C)}{1 - \tan B \tan C}.$

Hence $\quad \tan A + \tan B + \tan C = \tan A \tan B \tan C. \qquad (214)$

This relation leads to

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1. \qquad (215)$$

Exercises.—XIII.

In the identities 1–6, inclusive,

$$A + B + C = \pi.$$

1. Prove $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$ (216)

2. „ $\sin A + \sin B - \sin C = 4 \sin\dfrac{A}{2} \sin\dfrac{B}{2} \cos\dfrac{C}{2}.$

3. „ $\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0.$ (217)

4. „ $\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B) = 3 \sin A \sin B \sin C.$

5. „ $2 \sin^2 A \sin^2 B + 2 \sin^2 B \sin^2 C + 2 \sin^2 C \sin^2 A = \sin^4 A + \sin^4 B + \sin^4 C + 8 \sin A \sin B \sin C.$

6. „ $\sin^3 A + \sin^3 B + \sin^3 C = 3 \cos\dfrac{A}{2} \cos\dfrac{B}{2} \cos\dfrac{C}{2} + \cos\dfrac{3A}{2} \cos\dfrac{3B}{2} \cos\dfrac{3C}{2}.$

In the identities 7–10, inclusive,

$$a + b + c = 2s.$$

7. Prove $\Sigma \sin(s - a) \sin(s - b) + \sin s \Sigma \sin(s - a) = \Sigma \sin a \sin b.$

8. „ $\sin(s - a) \sin(s - b) + \sin s \,.\, \sin(s - c) = \sin a \sin b.$

9. Prove $\sin^2 s + \Sigma \sin^2(s - a) = 2(1 - \cos a \cos b \cos c)$.

10. ,, $\cos^2 s + \Sigma \cos^2(s - a) = 2(1 + \cos a \cos b \cos c)$.

In the identities 11–16, a, b, c are any three angles.

11. Prove $\Sigma \cos 2a \cos^2(b + c)$
$$= 2\cos(a + b)\cos(b + c)\cos(c + a) + \cos 2a \cos 2b \cos 2c.$$

12. ,, $\Sigma \cos^3 a \sin(b - c) = \cos(a + b + c)\sin(a - b)\sin(b - c)\sin(c - a)$.

13. ,, $\Sigma \sin 2a \sin^2(b + c)$
$$= 2\sin(a + b)\sin(b + c)\sin(c + a) + \sin 2a \sin 2b \sin 2c.$$

14. ,, $\Sigma \sin a \sin b \sin(a - b) = -\sin(a - b)\sin(b - c)\sin(c - a)$.

15. ,, $\Sigma \sin^3 a \sin^3(b - c)$
$$= 3\sin a \sin b \sin c \sin(a - b)\sin(b - c)\sin(c - a). \quad (218)$$

Dem.—Since $x + y + z$ is a factor in $x^3 + y^3 + z^3 - 3xyz$,

if $$x + y + z = 0, \quad x^3 + y^3 + z^3 = 3xyz.$$

Hence, putting
$$x = \sin a \sin(b - c), \quad y = \sin b \sin(c - a), \quad z = \sin c \sin(a - b),$$
the proposition is proved.

16. Prove $\Sigma \cos^3 a \sin^3(b - c)$
$$= 3\cos a \cos b \cos c \sin(a - b)\sin(b - c)\sin(c - a). \quad (219)$$

In the identities 17–21, inclusive,
$$\alpha + \beta + \gamma + \delta = 2\pi.$$

17. Prove $\cos\alpha + \cos\beta + \cos\gamma + \cos\delta + 4\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha + \gamma)$
$$\cos\tfrac{1}{2}(\alpha + \delta) = 0.$$

18. ,, $\cos\alpha - \cos\beta + \cos\gamma - \cos\delta - 4\sin\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha + \gamma)$
$$\sin\tfrac{1}{2}(\alpha + \delta) = 0.$$

19. ,, $\sin\alpha + \sin\beta + \sin\gamma + \sin\delta - 4\sin\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\alpha + \gamma)$
$$\sin\tfrac{1}{2}(\alpha + \delta) = 0.$$

20. ,, $\sin\alpha - \sin\beta + \sin\gamma - \sin\delta + 4\cos\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\alpha + \gamma)$
$$\cos\tfrac{1}{2}(\alpha + \delta) = 0.$$

21. ,, $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$
$$= \tan\alpha\tan\beta\tan\gamma + \tan\alpha\tan\beta\tan\delta + \tan\alpha\tan\gamma\tan\delta$$
$$+ \tan\beta\tan\gamma\tan\delta.$$

22. If a, b, c, d be any four angles, and if
$$a + b + c + d = 2s,$$
prove $$\sin a \sin b \sin c \sin d + \cos a \cos b \cos c \cos d$$
$$= \sin(s - a)\sin(s - b)\sin(s - c)\sin(s - d)$$
$$+ \cos(s - a)\cos(s - b)\cos(s - c)\cos(s - d). \quad (220)$$

23. From this theorem we may infer several others; thus, putting $d = 0$, we have

$$\cos a \cos b \cos c = \sin s \,.\, \sin(s-a)\sin(s-b)\sin(s-c) + \cos s \,.\, \cos(s-a)\cos(s-b)\cos(s-c). \qquad (221)$$

24. Prove $\cos(b+c)\cos(c+a)\cos(a+b) = \sin a \sin b \sin c \sin(a+b+c) + \cos a \cos b \cos c \cos(a+b+c)$. (222)

25. Prove $\sin^3(x-a)\sin^3(b-c)+\sin^3(x-b)\sin^3(c-a)+\sin^3(x-c)\sin^3(a-b)$ $=3\sin(x-a)\sin(x-b)\sin(x-c)\sin(a-b)\sin(b-c)\sin(c-a)$.

26. If A, B, C be the angles of a triangle, prove that

$$\left(\frac{\sin(B-C)}{\sin A}+\frac{\sin(C-A)}{\sin B}+\frac{\sin(A-B)}{\sin C}\right)$$
$$\left(\frac{\sin A}{\sin(B-C)}+\frac{\sin B}{\sin(C-A)}+\frac{\sin C}{(\sin A-B)}\right)$$
$$= 9 - 4(\sin^2 A + \sin^2 B + \sin^2 C). \qquad (223)$$

27. In the same case prove

$$\Sigma \sin A \sin^3(B-C) = 4\sin A \sin B \sin C \,.\, \sin(A-B)\sin(B-C)\sin(C-A). \qquad (224)$$

28. Prove $\Sigma \sin A \cos^2 A \sin^3(B-C)$ $= \sin A \sin B \sin C \sin(A-B)\sin(B-C)\sin C - A)$. (225)

29. For any three angles, prove that

$$\cos\alpha\cos\beta\sin(\alpha-\beta)+\cos\beta\cos\gamma\sin(\beta-\gamma)+\cos\gamma\cos\alpha\sin(\gamma-\alpha) + \sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha) = 0.$$

30. Prove that in any triangle

$$(\cos A \sin^3 B - \cos B \sin^3 A)/\sin(A-B) = 1 + \cos A \cos B \cos C.$$

31. Prove $\Sigma \sin^5 A \sin(B-C)$ $= -\sin A \sin B \sin C \sin(A-B)\sin(B-C)\sin(C-A)$.

32. Prove $$\Sigma\left(\frac{\cot A + \cot B}{\cot\frac{A}{2}+\cot\frac{B}{2}}\right) = 1.$$

CHAPTER III.

THEORY OF LOGARITHMS.

SECTION I.—PRELIMINARY PROPOSITIONS.

73.—DEF. I.—*If a variable quantity x approach a fixed quantity a, so as to differ from it by less than any assignable quantity however small, but never actually reach it, the fixed quantity is called the limit of the variable one.* Thus, the limit of the sum of n terms of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c., when n increases without limit, is unity.

74. *The limit of the sum of a finite number of variable quantities x, y, z, &c., is equal to the sum of the limits of these quantities.*

For, if a, b, c, &c., be the limits of x, y, z, &c., we have

$$(x + y + z + \&c.) - (a + b + c + \&c.) = (x - a) + (y - b) + (z - c) + \&c$$

Now, if n denote the number of quantities, and ϵ denote a quantity as small as we please,

$$(x - a), \quad (y - b), \quad (z - c), \quad \&c.,$$

can each by hypothesis become less than ϵ/n. Therefore

$$(x + y + z + \&c.) - (a + b + c + \&c.),$$

can become less than ϵ. Hence the proposition is proved.

This theorem may not hold when the number of the quantities becomes infinite. Thus, if n be a positive quantity,

$$\frac{n}{n+1} > \frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{2n} > \frac{n}{2n}.$$

When n increases indefinitely, each term of the sum

$$\frac{1}{n+1} + \frac{1}{n+2} \cdots \frac{1}{2n}$$

has zero for limit; but the sum is comprised between $\frac{1}{2}$ and 1.

75. *The limit of the product of a finite number of variable quantities x, y, z, &c., is equal to the product of the limits of these quantities.*

For, let a, b, c, &c., be the limits of x, y, z, we can put

$$x = a + \alpha, \quad y = b + \beta, \text{ \&c.,}$$

α, β. &c., being as small as we please.

Hence $$xyz, \text{ \&c.} - abc, \text{ \&c.} = \alpha\beta\gamma + a\beta\gamma + b\gamma\alpha \ldots$$

In the last quantity each term has zero for limit, and the number of terms is finite. Hence the limit of

$$xyz, \text{ \&c.} - abc, \text{ \&c., is } 0.$$

If the number of factors increase without limit, this principle may not be valid. Thus, for example,

$$\frac{n}{n+1} \cdot \frac{n+1}{n+2} \cdot \frac{n+2}{n+3} \cdots \frac{2n-1}{2n} = \frac{1}{2}.$$

When n increases indefinitely, each factor has unity for limit; but the limit of the product remains equal to $\frac{1}{2}$.

76. Def. II.—*A series is said to be convergent when the sum of its first n terms tends towards a limit according as n increases. This limit is called the sum of the series.*

Example.—A decreasing geometrical progression

$$a + ar + ar^2 + \ldots$$

is a convergent series, for the sum of the first n terms

$$= \frac{a}{1-r} - \frac{ar^n}{1-r};$$

but the limit of $$\frac{ar^n}{1-r} = 0;$$

therefore limit of sum $$= \frac{a}{1-r}.$$

Remark.—A series not convergent is said to be *divergent.* "A divergent series can never be equal to a determinate quantity. It is merely an expression possessing certain properties relating to operations to which the series is submitted."—Abel.

Cor.—A series consisting of the sum or the difference of two converging series is convergent.

77. *A series is convergent when the ratio of any term to the preceding term is less in absolute value than a fixed quantity is less than unity.*

Let the series to positive terms be

$$u_1 + u_2 + u_3 + \&c.,$$

such that
$$\frac{u_{n+1}}{u_n} < r < 1;$$

then $u_{n+1} < ru_n$, $u_{n+2} < ru_{n+1} < r^2 u_n$, $u_{n+3} < ru_{n+2} < r^3 u_n \ldots$

Therefore the sum $u_n + u_{n+1} + u_{n+2} \ldots u_{n+p}$,

which increases with p, remains less than the limit of the progression
$$u_n + ru_n + r^2 u_n + \&c., \quad \text{or} \quad \frac{u_n}{1-r}.$$

Thus the sum tends towards a limit, and the series is convergent.

Cor.—If the series has some of its terms negative, it will be the difference between two convergent series, and therefore will be convergent.

Section II.—The Exponential Theorem.

78. Lemma.—*If a be a positive number different from unity, the function a^x is continuous.*

We shall first demonstrate the theorem for a number $a = 1 + b$, greater than unity.

1°. If x proceeds by successive integer values,

$$0, 1, 2, 3, \ldots a^x$$

increases, and becomes as great as we please.

In fact,

$$(1+b)^{n+1} = (1+b)^n(1+b) = (1+b)^n + b(1+b)^n > (1+b)^n;$$

therefore $(1+b)^n$ increases with n. In order that

$$(1+b)^n > A,$$

or

$$1 + nb + \frac{n \cdot n-1}{\lfloor 2} b^2 + \&c. > A,$$

it suffices that $1 + nb = A$, or $n = \dfrac{A-1}{b}$.

2°. If x proceeds by the successive integers 2, 3, &c., $\sqrt[x]{a}$ decreases, and approaches as near as we please to unity.

Since a is greater than unity, $\sqrt[x]{a}$ cannot be < 1. Then let ϵ be a small quantity. We can choose x in such a manner that

$$\sqrt[x]{a} < 1 + \epsilon; \quad \text{for then} \quad (1+\epsilon)^x > a,$$

or

$$1 + x\epsilon + \frac{x \cdot x-1}{\lfloor 2} \epsilon^2 + \&c. > a.$$

It suffices to have

$$x = \frac{a-1}{\epsilon}.$$

3°. Giving now to x positive fractional values (including improper fractions), I say a^x increases with x, and increases by degrees as small as we please. In fact, at first let

$$x = \frac{m}{n}, \quad \text{then} \quad a^x = a^{\frac{m}{n}};$$

but

$$a^m > 1; \quad \therefore \quad a^{\frac{m}{n}} > 1.$$

Again, let

$$x = m + \frac{1}{n},$$

then

$$a^{m+\frac{1}{n}} - a^m = a^m\left(a^{\frac{1}{n}} - 1\right);$$

but if we wish to have

$$a^{m+\frac{1}{n}} - a^m < \epsilon,$$

we can place

$$a^m\left(a^{\frac{1}{n}} - 1\right) < \epsilon, \quad \text{or} \quad \left(a^{\frac{1}{n}} < \frac{\epsilon}{a^m} + 1\right),$$

or

$$\left(1 + \frac{\epsilon}{a^m}\right)^n > a.$$

This condition will be fulfilled if

$$1 + \frac{n\epsilon}{a^m} > a, \quad \text{or} \quad n > \frac{(a-1)a^m}{\epsilon}.$$

Hence it follows, that if x varies by insensible degrees from 0 *to* $+\infty$, *a^x varies by insensible degrees from* 1 *to* $+\infty$. Since $a^{-x} = \frac{1}{a^x}$, we see that when x varies from 0 to $-\infty$, a^x varies from 1 to 0.

If a be < 1, we consider $\frac{1}{a'^x}$, after having put $a = \frac{1}{a'}$.

79. Limit of $\left(1 + \frac{1}{n}\right)^n$ for $n = \infty$.

The Binomial Theorem gives

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n \,.\, n-1}{1 \,.\, 2} \,.\, \left(\frac{1}{n}\right)^2 + \frac{n \,.\, n-1 \,.\, n-2}{1 \,.\, 2 \,.\, 3}\left(\frac{1}{n}\right)^3 + \&c.$$

$$= 1 + \frac{1}{1} + \frac{1 - \frac{1}{n}}{1 \,.\, 2} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{1 \,.\, 2 \,.\, 3} + \ldots \quad (A).$$

When n increases indefinitely, the fractions $\frac{1}{n}, \frac{2}{n} \ldots$ become indefinitely small, and have zero for limit. This leads to the conjecture that the limit of

$$\left(1 + \frac{1}{n}\right)^n \text{ is } = 1 + 1 + \frac{1}{1 \,.\, 2} + \frac{1}{1 \,.\, 2 \,.\, 3} + \&c.$$

We shall now prove rigorously that it is.*

(*a*) The series

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots \qquad (B)$$

is convergent. For starting from the 4th terms, the terms are less than those of the decreasing geometrical progression

$$\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}, \text{ \&c.},$$

and this tends towards a limit.

We shall denote the sum of the series (B) by e.

(*b*) The second member of (A) tends towards a limit equal to or less than e. In fact, the terms of the second number of (A) are less than the corresponding terms of (B). These terms increase with n; their number also increases with n; therefore

$$\left(1+\frac{1}{n}\right)^n$$

increases with n, but cannot exceed e. Then it has a limit equal to or inferior to e. We shall denote this limit by ϵ.

(*c*) Putting

$$e_p = 1+\frac{1}{1}+\frac{1}{1.2}+\text{\&c.}\ldots\frac{1}{1.2.3\ldots p},$$

$$\epsilon_p = 1+\frac{1}{1}+\frac{1-\frac{1}{n}}{1.2}+\ldots\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{p-1}{n}\right)}{1.2.3\ldots p}.$$

Let η be a quantity as small as we please, since e_p has e as its limit when p increases indefinitely; we can therefore take for p

* This is only a conjecture, unless proved. For the principles, §§ 74, 75, are inapplicable, except when the number of terms of a sum or product remains *finite*.

a value sufficiently great to make

$$e - e_p < \frac{\eta}{2}.$$

Now, leaving this value of p FIXED, and making n increase indefinitely, the terms of ϵ_p will have for limits the corresponding terms of e_p. Then we can take n sufficiently great for

$$e_p - \epsilon_p < \frac{\eta}{2}.$$

Hence, adding the two inequalities,

$$e - e_p < \frac{\eta}{2}, \quad \text{and} \quad e_p - \epsilon_p < \frac{\eta}{2},$$

we get $$e - \epsilon_p < \eta;$$

but limit of ϵ_p is $$\left(1 + \frac{1}{n}\right)^n \text{ or } \epsilon.$$

Hence $e - \epsilon < \eta$; but η may be as small as we please; therefore

the limit of $$\left(1 + \frac{1}{n}\right)^n \text{ is } e. \tag{226}$$

This equality still subsists, if n approaches infinity by fractional values; for example, if n be of the form $\frac{p}{q}$, and p proceeds towards infinity by integer values. For if n be comprised between the integer numbers m and $m + 1$, we have

$$\left(1 + \frac{1}{m}\right)^{m+1} > \left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{m + 1}\right)^m,$$

or

$$\left(1 + \frac{1}{m}\right)^m \left(1 + \frac{1}{m}\right) > \left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{m + 1}\right)^{m+1} \div \left(1 + \frac{1}{m + 1}\right).$$

If m increase indefinitely, the first member and the last have

the common limit e; therefore the limit of

$$\left(1+\frac{1}{n}\right)^n \text{ is } e.$$

Again, if $n = -m$, we have

$$\left(1+\frac{1}{n}\right)^n = \frac{1}{\left(1-\frac{1}{m}\right)^m} = \left(\frac{m}{m-1}\right)^m = \left(1+\frac{1}{m-1}\right)^m$$

$$= \left(1+\frac{1}{m-1}\right)^{m-1} \times \left(1+\frac{1}{m-1}\right);$$

but the limit of

$$\left(1+\frac{1}{m-1}\right)^{m-1},$$

when m approaches infinity, is e. Hence the proposition holds for infinitely negative values of n.

This theorem is fundamental; its demonstration is imperfect nearly in every English mathematical work that the author is acquainted with.

80. If we put $e = 1 + \frac{1}{1} + \frac{1}{\lfloor 2} + \ldots \frac{1}{\lfloor n} + \rho,$

the value of ρ is $$< \frac{1}{n\lfloor n}.$$

Dem.—By hypothesis,

$$\rho = \frac{1}{\lfloor n}\left\{\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \text{\&c. to inf.}\right\}.$$

Hence ρ is $< \dfrac{1}{\lfloor n}\left\{\dfrac{1}{n+1} + \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+1)^3} + \text{\&c. to inf.}\right\};$

that is $$< \frac{1}{\lfloor n}\cdot\frac{1}{n}, \quad \text{or} \quad \frac{1}{n\lfloor n}. \qquad (227)$$

Cor. 1.—If in the numerical calculation of the value of e we omit all the terms after the $(n+1)^{th}$, we commit an error less than the n^{th} part of that term.

Cor. 2.—Calculation of e.

We have $$e = 1 + \frac{1}{1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} \cdots$$

$$1 + \frac{1}{1} = 2{\cdot}000000000,$$

$$\frac{1}{\lfloor 2} = 0{\cdot}500000000,$$

$$\frac{1}{\lfloor 3} = 0{\cdot}166666666,$$

$$\frac{1}{\lfloor 4} = 0{\cdot}041666666,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{1}{\lfloor 12} = 0{\cdot}000000002;$$

$$\therefore \quad e = 2{\cdot}718281823.$$

This value has two errors:—1°. We have taken 9 places of decimals, and each term after the two first has an error less than 1 unit of the 9th order, which makes for 10 terms an error which is less than 10 units of the 9th order. 2°. The error in omitting all terms after $\frac{1}{\lfloor 12}$ is less than $\frac{1}{12}$ of $\frac{1}{\lfloor 12}$; that is, an error which is less than 1 unit of the 9th order. Hence the total error is less than 11 units of the 9th order. Hence $e = 2{\cdot}7182818$ with 7 decimal places exact.

Cor. 3.—*The value of e is incommensurable.*

Dem.—If possible, let $e = \dfrac{m}{n}$, where m and n are integers.

Then $$\frac{m}{n} = 2 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \ldots \frac{1}{\lfloor\underline{n}} + \frac{1}{\lfloor\underline{n+1}} + \frac{1}{\lfloor\underline{n+2}} + \&c.$$

Hence $$m\lfloor\underline{n-1} = \text{an integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.$$

The sum of the fractions on the right is evidently $> \dfrac{1}{n+1}$, and by the demonstration of § 80, $< \dfrac{1}{n}$. Thus a quantity which lies between the fractions $\dfrac{1}{n}$, $\dfrac{1}{n+1}$, is equal to an integer, which is absurd. *Hence e is incommensurable.*

81. The sum of the infinite series

$$1 + x + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} + \frac{x^4}{\lfloor\underline{4}} + \&c. \text{ is } e^x.$$

Dem.—Raising both sides of the equation,

$$\text{the limit of } \left(1 + \frac{1}{n}\right)^n = e$$

to the power x, we get

$$\text{the limit of } \left(1 + \frac{1}{n}\right)^{nx} = e^x.$$

And it may be proved, as in § 79, that

$$\text{the limit of } \left(1 + \frac{1}{n}\right)^{nx},$$

when n becomes indefinitely large,

$$= 1 + x + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} + \&c.$$

Hence $$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{\&c., to infinity.} \qquad (228)$$

This result, which is called the exponential theorem, is one of the most important in the whole range of Mathematics.

Cor. 1.—The limit of $\left(1 + \frac{x}{n}\right)^n$ is e^x. (229)

Exercises.—XIV.

1. Prove $1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \frac{1}{\lfloor 7}$, &c. to infinity $= \frac{1}{2}(e - e^{-1})$.

2. ,, $x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} +$ &c. ,, $= \frac{1}{2}(e^x - e^{-x})$.

3. ,, $1 + \frac{x^2}{\lfloor 2} + \frac{x^2}{\lfloor 4} +$ &c. ,, $= \frac{1}{2}(e^x + e^{-x})$.

4. ,, $\frac{1}{\lfloor 2} + \frac{1+2}{\lfloor 3} + \frac{1+2+3}{\lfloor 4}$, &c. to infinity $= \frac{e}{2}$.

5. Find the sum of $\frac{1}{\lfloor 1} + \frac{2}{\lfloor 3} + \frac{3}{\lfloor 5} +$ &c. to infinity.

6. ,, $\frac{1}{\lfloor 3} + \frac{2}{\lfloor 5} + \frac{3}{\lfloor 7}$, &c. to infinity.

7. ,, $1 + \frac{1+x}{\lfloor 2} + \frac{1+x+x^2}{\lfloor 3} + \frac{1+x+x^2+x^3}{\lfloor 4}$, &c.

Section III.—Napierian Logarithms.

82. Def. I.—*The logarithm of a number is the power to which another given number, called the* base, *must be raised in order to produce it.*

Thus, if $e^x = n$, x is the logarithm (contracted log) of n to the base e, and is written $x = \log_e n$, the base being put as a suffix.

Again, if $$10^y = n, \quad y = \log_{10} n.$$

Def. II.—*Logarithms to the base e* (§ 80, *Cor.* 2) *are called* Napierian *Logarithms, and those to the base* 10, Common *logarithms.*

Theoretically, any number may be taken as base, but only two are in use, namely, the Napierian base e (called after Napier, the inventor of logarithms), and the common base 10. Common logarithms are employed in practical calculations, such as Geodesy, Navigation, &c., and Napierian logarithms in Analytical Mathematics.

83. Fundamental Properties.—

1°. *The log of* 1 *to any base is zero.*

For $$a^0 = 1\,; \quad \therefore\ \log_a 1 = 0. \tag{230}$$

2°. *The log of the base is unity.*

For $$a^1 = a\,; \quad \therefore\ \log_a a = 1.$$

3°. *The log of the product of any two factors is equal to the sum of the logs of the factors.*

For if $$a^x = m, \quad x = \log_n m,$$

and $$a^y = n, \quad y = \log_a n\,;$$

therefore $$a^{x+y} = mn \text{ and } x + y = \log_a mn\,;$$

but $$x + y = \log_a m + \log_a n.$$

Hence $$\log_a mn = \log_a m + \log_a n. \tag{231}$$

And so on for any greater number of factors.

4°. *The log of the quotient of two numbers is equal to the difference of their logs.*

For since the product of the divisor and quotient is equal to the dividend, the sum of their logs is equal to the log of the dividend;

$$\therefore\ \text{log quotient} = \text{log dividend} - \text{log divisor}. \tag{232}$$

5°. *The log of any power (integral or fractional, positive or negative) of a number is equal to the log of the number multiplied by the index of the power.*

For if $a^x = m$, $x = \log_a m$; then $a^{rx} = m^r$, and $rx = \log_a m^r$.

Hence
$$\log_a m^r = r \log_a m. \tag{233}$$

6°. *If a series of terms be in G. P., their logs are in A. P.*

Let the *GP* be $a, ab, ab^2, ab^3 \ldots$

The logs are (3°, 5°)—

$$\log a,\ \log a + \log b,\ \log a + 2 \log b,\ \log a + 3 \log b,\ \&c.,$$

which are in *A. P.*, the common difference being $\log b$.

84. Expansion of $log_e(1 + x)$.

It will be sufficient to consider the case where x is not greater than unity, as it will be found that all others can be reduced to it.

Let $z = \log_e(1 + x)$; then $e^z = 1 + x$;

but $e^z = \text{limit of } \left(1 + \frac{z}{n}\right)^n$. (§ 81, *Cor.* 1.)

Hence

$$z = \text{limit of } x + \left(\frac{1}{n} - 1\right)\frac{x^2}{\underline{|2}} + \left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\frac{x^3}{\underline{|3}} \ldots\ldots$$

$$= \text{limit of } x - \left(1 - \frac{1}{n}\right)\frac{x^2}{2}$$

$$+ \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)\frac{x^3}{3} \ldots \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right) - \left(1 - \frac{1}{pn}\right)\frac{x^{p+1}}{p+1} + \&c.$$

Now let R_p denote the sum of all the terms on the right after the p^{th}; then

$$R_p = (-1)^p \left(1 - \frac{1}{n}\right) \ldots \left(1 - \frac{1}{pn}\right)\frac{x^{p+1}}{p+1}$$

$$\times \left\{1 - \frac{p + 1 - \frac{1}{n}}{p+2}x + \frac{\left(p + 1 - \frac{1}{n}\right)\left(p + 2 - \frac{1}{n}\right)}{(p+2)(p+3)}x^2 + \&c.\right\}$$

Now the series within the parentheses will be $< 1 + r + r^2 + r^3$, &c., to infinity, where r is less than unity, because x is not greater than unity, and the coefficient of each term bears to that of the preceding a ratio which is less than unity. Hence θ denoting some proper fraction, the series within the parentheses can be denoted by $\frac{\theta}{1-r}$. Therefore

$$z = \text{limit of } x - \left(1 - \frac{1}{n}\right)\frac{x^2}{2} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)\frac{x^3}{3} \ldots$$

$$+ (-1)^{p-1}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)\ldots\left(1 - \frac{1}{(p-1)n}\right)\frac{x^p}{p}$$

$$+ (-1)^p\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)\ldots\left(1 - \frac{1}{pn}\right)\frac{x^{p+1}}{p+1}\cdot\frac{\theta}{1-r}.$$

Hence, p remaining finite, and making n infinite, we have

$$z = x - \frac{x^2}{2} + \frac{x^3}{3} - \text{\&c.} \ldots (-1)^p.\frac{x^{p+1}}{p+1}.\frac{\theta}{1-r}. \quad (234)$$

Now, since $\frac{x^{p+1}}{p+1}$ tends towards zero as p tends towards infinity, we have

$$z \text{ or } \log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots \text{ to infinity.} \quad (235)$$

Cor.— $$\text{Log}_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots. \quad (236)$$

85. If in equation (235) we change x into $-x$, we get

$$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots;$$

$$\therefore \log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{\&c.}\right) \quad (237)$$

Hence, putting $x = \frac{1}{2n+1}$ or $\frac{1+x}{1-x} = \frac{n+1}{n}$, we get

$$\log_e\left(\frac{n+1}{n}\right) = 2\left\{\frac{1}{(2n+1)} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \&c.\right\},$$

or

$$\log_e(n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1^3)} + \frac{1}{5(2n+1)^5} + \&c.\right\} \quad (238)$$

Again, in (237), putting $x = \dfrac{1}{2n^2-1}$, and we get

$$\log_e(n+1) - \log_e n = \log_e n - \log_e(n-1)$$
$$-2\left\{\frac{1}{2n^2-1} + \frac{1}{3(2n^2-1)^3} + \frac{1}{5(2n^2-1)^5} + \&c.\right\} \quad (239)$$

This formula is important in the calculation of successive tabular differences.

Exercises.—XV.

1. Prove $\log_e x = 2\left\{\dfrac{x-1}{x+1} + \dfrac{1}{3}\left(\dfrac{x-1}{x+1}\right)^3 + \&c.\right\}$. (240)

2. ,, $\log_e x = \log_e a + \dfrac{x-a}{a} - \dfrac{1}{2}\left(\dfrac{x-a}{a}\right)^2 + \dfrac{1}{3}\left(\dfrac{x-a}{a}\right)^3$. (241)

3. ,, $\log_e(h+n) = \log_e n + \dfrac{h}{n} - \dfrac{1}{2}\left(\dfrac{h}{n}\right)^2 + \dfrac{1}{3}\left(\dfrac{h}{n}\right)^3 + \&c.$ (242)

4. ,, $\log_e(n+h) = \log_e n + 2\left\{\dfrac{h}{2n+h} + \dfrac{1}{3}\left(\dfrac{h}{2n+h}\right)^3 + \&c.\right\}$ (243)

5. ,, $\log_e \sec x = \frac{1}{2}\sin^2 x + \frac{1}{4}\sin^4 x + \frac{1}{6}\sin^6 x + \&c.$ (244)

6. ,, $\log_e \sec x = \frac{1}{2}\tan^2 x - \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + \&c.$ (245)

7. ,, $\log_e x + 1 = \log_e x + \dfrac{1}{x} - \dfrac{1}{2x^2} + \dfrac{1}{3x^3} - \&c.$ (246)

8. ,, $\log_e \sqrt{2} = \dfrac{1}{4}\left\{\dfrac{1}{1.1} + \dfrac{1}{2.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \&c.\right.$ (247)

9. ,, $a^x = 1 + (\log_e a)x + (\log_e a)^2\dfrac{x^2}{\underline{|2}} + (\log_e a)^3\dfrac{x^3}{\underline{|3}} + \&c.$ (248)

Put $a^x = e^z$, and taking logs, we get $z = (\log_e a).x$;

but $e^z = 1 + z + \dfrac{z^2}{\underline{|2}} + \dfrac{z^3}{\underline{|3}} + \&c.$ Hence, &c.

10. Prove the limit of $\cos^n\left(\frac{a}{n}\right)$, when n increases without limit, is 1.

Put $$u = \cos^n\left(\frac{a}{n}\right), \text{ or } u = \left(1 - \sin^2\frac{a}{n}\right)^{\frac{n}{2}};$$

therefore $$\log_e u = \frac{n}{2}\log_e\left(1 - \sin^2\frac{a}{n}\right).$$

Hence $$\log_e u = -\frac{n}{2}\left\{\sin^2\frac{a}{n} + \frac{1}{2}\sin^4\frac{a}{n} + \frac{1}{3}\sin^6\frac{a}{n} + \&c.\right\};$$

but when n increases without limit,

$$\sin\frac{a}{n} = \frac{a}{n},$$

the omitted terms being infinitesimals of the third and higher orders. Hence, when n is infinite,

$$\log_e u = -\frac{n}{2}\left\{\frac{a^2}{n^2} + \frac{1}{2}\frac{a^4}{n^4} + \frac{1}{3}\frac{a^6}{n^6}, \ \&c.\right\}$$

$$= -\left\{\frac{1}{2}\frac{a^2}{n} + \frac{1}{4}\frac{a^4}{n^3} + \frac{1}{6}\frac{a^6}{n^5} + \&c.\right\} = 0;$$

$$\therefore \quad u = 1; \text{ that is, limit of } \cos^n\left(\frac{a}{n}\right) = 1.$$

11. Prove, in the same case, the limit of

$$\left(\cos\frac{a}{n}\right)^{n^2} \text{ is } e^{-\frac{a^2}{2}}.$$

12. Prove, in the same case, the limit of

$$\left(\cos\frac{a}{n}\right)^{n^3} \text{ is infinite.}$$

13. Prove $\log_e \sin 2\theta + \log_e \tan\theta + \cos 2\theta + \frac{1}{2}\cos^2 2\theta + \frac{1}{3}\cos^3 2\theta + \&c.$, to infinity. $= 0$

14. If a, b, c be consecutive integers; prove that

$$\log_e b = \tfrac{1}{2}\log_e ac + \frac{1}{1 + 2ac} + \frac{1}{3(1 + 2ac)^3} + \frac{1}{5(1 + 2ac)^5} + \&c. \qquad (249)$$

15. If a_1, a_2 be the roots of the equation $x^2 + px + q = 0$; prove that

$$\log_e(1 - px + qx^2) = (a_1 + a_2)x - \tfrac{1}{2}(a_1^2 + a_2^2)x^2 + \tfrac{1}{3}(a_1^3 + a_2^3)x^3 - \&c.$$

16. Find the sum of (250)

$$(1 + 3)\log_e 3 + \frac{(1 + 3^2)(\log_e 3)^2}{\underline{|2}} + \frac{(1 + 3^3)(\log_e 3)^3}{\underline{|3}} + \&c.$$

17. Prove $(1 + x)\log(1 + x) = x + \dfrac{x^2}{1\,.\,2} - \dfrac{x^3}{2\,.\,3} + \dfrac{x^4}{3\,.\,4}$, &c.

86. Calculation of Napierian Logarithms.

1°. *To calculate* $\log_e 2$. In equation (238) put $n = 1$, and we get

$$\log_e 2 = 2\left\{\tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{1}{3}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{3}\right)^5 + \tfrac{1}{7}\left(\tfrac{1}{3}\right)^7 + \text{\&c.}\right\}.$$

The value of this series correct to 7 places of decimals is found as follows:—

$\frac{1}{3}$ $= \cdot33333333.$	Hence	$\frac{1}{3}$	$= \cdot33333333.$
$(\frac{1}{3})^3 = \cdot03703704.$	,,	$\frac{1}{3}(\frac{1}{3})^3$	$= \cdot01234568.$
$(\frac{1}{3})^5 = \cdot00411523.$	,,	$\frac{1}{5}(\frac{1}{3})^5$	$= \cdot00082305.$
$(\frac{1}{3})^7 = \cdot00045725.$	,,	$\frac{1}{7}(\frac{1}{3})^7$	$= \cdot00006532.$
$(\frac{1}{3})^9 = \cdot00005080.$	,,	$\frac{1}{9}(\frac{1}{3})^9$	$= \cdot00000564.$
$(\frac{1}{3})^{11} = \cdot00000564.$	,,	$\frac{1}{11}(\frac{1}{3})^{11}$	$= \cdot00000051.$
$(\frac{1}{3})^{13} = \cdot00000063.$	,,	$\frac{1}{13}(\frac{1}{3})^{13}$	$= \cdot00000005.$
		sum	$= \cdot34657358$

Hence $\log_e = \cdot69314716.$ (251)

In the foregoing calculation every term has an error, which is less than 1 unit of the 8th order, and it will be perceived that some are plus and some are minus. Hence the total error is less than 7 units of the 8th order; and, therefore, the result must be true for 7 decimal places.

Cor.—Since $\log_e 2^n = n \log_e 2$, equation (233), the log of any power of 2 is got by multiplying $\log_e 2$ by the index of the power. Thus, $\log_e 8$ is equal to $\cdot69314716$ multiplied by 3, &c. Hence, having calculated $\log_e 2$, we have virtually calculated the Napierian logarithms of all the powers of 2.

2°. *To calculate* $\log_e 3$. In (238) put $n = 2$, and we get

$$\log_e 3 = \log_e 2 + 2\left\{\tfrac{1}{5} + \tfrac{1}{3}\left(\tfrac{1}{5}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{5}\right)^5 + \text{\&c.}\right\},$$

which is calculated thus:

$\frac{1}{5} = \cdot 20000000.$	Hence	$\frac{1}{5}$	$= \cdot 20000000.$
$(\frac{1}{5})^3 = \cdot 00800000.$	,,	$\frac{1}{3}(\frac{1}{5})^3$	$= \cdot 00266667.$
$(\frac{1}{5})^5 = \cdot 00032000.$	,,	$\frac{1}{5}(\frac{1}{5})^5$	$= \cdot 00006400.$
$(\frac{1}{5})^7 = \cdot 00001280.$	,,	$\frac{1}{7}(\frac{1}{5})^7$	$= \cdot 00000183.$
$(\frac{1}{5})^9 = \cdot 00000051.$	,,	$\frac{1}{9}(\frac{1}{5})^9$	$= \cdot 00000006.$
		sum	$=$ 20273256

Hence $\log_e 3 = 1\cdot 09861228.$

Having found $\log_e 3$, we get the logs of 3^2, 3^3, 3^4, &c., by multiplying it by the indices 2, 3, 4, &c. Hence the Napierian logs of all powers of 3 are found.

3°. *To calculate $\log_e 5$.* In (238) put $n = 4$, and we get

$$\log_e 5 = \log_e 4 + 2\left\{\tfrac{1}{9} + \tfrac{1}{3}(\tfrac{1}{9})^3 + \tfrac{1}{5}(\tfrac{1}{9})^5 + \&c.\right\}.$$

The terms within the parentheses are summed as in 1° and 2°; and since $\log_e 4 = 2\log_e 2 = 1\cdot 38629432$, we get

$$\log_e 5 = 1\cdot 60943793. \qquad (253)$$

4°. From these examples the method of calculating the Napierian logarithms of all prime numbers is obvious. The logarithms of numbers which are the products of primes are found by adding the logarithms of the factors, § 83, 3°. Thus $\log_e 35$ is found by adding $\log_e 5$ and $\log_e 7$. Hence the mode of calculating the logarithms of all numbers is evident.

Observation.—In equation (238) the larger n is the more rapidly, the terms of the series within the parentheses converge. It will be seen that when n is large, it will not in the summation be necessary to take more than two terms into account, and when n exceeds 500, that one term will suffice in the calculation if taken only to seven places of decimals.

SECTION IV.—COMMON LOGARITHMS.

87. Connection between Napierian and Common Logarithms.—Let x, y denote the Napierian and the common logarithm respectively of any number n; then we have

$$e^x = n, \quad 10^y = n.$$

Hence $$e^x = 10^y; \quad \therefore \; e^{\frac{x}{y}} = 10.$$

Hence $$\frac{x}{y} = \log_e 10 = 2{\cdot}30258509;$$

therefore $$\frac{y}{x} = \frac{1}{2{\cdot}30258509} = {\cdot}43429448.$$

The number just found, viz. ·43429448 is called the modulus of the common system of logarithms. It is usually denoted by μ.

Hence $$y = \mu x. \tag{254}$$

Therefore, if the Napierian logarithms of any number be multiplied by μ, that is, by ·43429448, the product will be the common logarithm.

Cor. 1.—*The modulus is the reciprocal of the Napierian log of* 10.

Cor. 2.—*In the same manner the modulus of the logarithms for any base is the reciprocal of the Napierian logarithm of that base.*

88. Advantages of the Common System of Logarithms.

DEF.—*If the log of a number consist of an integer and a decimal, the integer is called the* CHARACTERISTIC *of the logarithm, and the decimal the* MANTISSA. Thus, $\log_{10} 1728 = 3{\cdot}2375439$, the characteristic of which is 3, and the mantissa ·2375439.

1°. *In the common system, the characteristic of the log of any number greater than unity is one less than the number of figures to the left of the decimal point.*

DEM.—If there be $n + 1$ figures to the left of the decimal

point, the number lies between 10^n and 10^{n+1}. Hence its log lies between n and $n + 1$: in other words, the log is n and a decimal. Hence the characteristic is n. Thus the log of 558·36 is 2·7469143, whose characteristic is 2.

2°. *The log of a decimal commencing with cyphers has a negative characteristic, and is arithmetically one greater than the number of cyphers.*

DEM.—If there be n cyphers between the decimal point and the first significant figure, the value of the decimal lies between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is, between 10^{-n} and $10^{-(n+1)}$. Hence the log will lie between $-n$ and $-(n + 1)$, and we make the convention that the mantissa will be always positive. The characteristic will be $-(n + 1)$. Thus, log $\cdot 0067836 = \bar{3}\cdot 8314602$. Here it is to be remarked that when the characteristic of a log is negative, the sign minus is always placed over, as in the present example. The reason is, if the sign were placed in the usual manner before the log, it would mean that both characteristic and mantissa were negative.

The present proposition implies the following:—*The logarithm of any number* < 1 *has a negative characteristic.*

EXAMPLE.—

$$\log \frac{852}{27354} = \log 852 - \log 27354 = \begin{array}{r} 2\cdot 9304396 \\ 4\cdot 4370208 \\ \hline \bar{2}\cdot 4934088 \end{array}$$

We subtract the mantissa in the usual way; but when we come to the characteristic, we say 4 from 2 leaves -2, which we write $\bar{2}$.

3°. *If two numbers consist of the same digits, but differ in the position of the decimal point, their logarithms have the same mantissa.*

For, if the numbers differ only in the position of the decimal point, the greater must be equal to the less, multiplied by some power of ten. Hence their logarithms can differ only in their characteristics, and therefore must have the same mantissa.

89. Proportional Parts.—*If n, $n+h$, $n+h'$, be three numbers, whose differences are small compared to n, then approximately*

$$log_{10}(n+h) - log_{10}n : log_{10}(n+h') - log_{10}n :: h : h'. \qquad (255)$$

DEM.—

$$\log_{10}(n+h) - \log_{10}n = \log_{10}\left(1+\frac{h}{n}\right) = \mu \log_e\left(1+\frac{h}{n}\right), \ \S\ 87;$$

$$\therefore \quad \log_{10}(n+h) - \log_{10}n = \mu\left(\frac{h}{n} - \frac{1}{2}\frac{h^2}{n^2} + \frac{1}{3}\frac{h^3}{n^3} - \&c.\right).$$

Now, suppose n is an integer containing at least five figures, and that h is not greater than unity, then $\frac{h}{n}$ is not greater than $\cdot 0001$; and since $\mu = \cdot 43429448$, the second term on the right-hand side in the preceding equation is not greater than $\cdot 0000000021714724$, and the third term is less than the ten-thousandth part of this. Hence, if $\frac{h}{n}$ be not greater than $\cdot 0001$,

$$\log_{10}(n+h) - \log_{10}n = \frac{\mu h}{n}$$

correct to eight places of decimals. In like manner,

$$\log_{10}(n+h') - \log_{10}n = \frac{\mu h'}{n}.$$

Hence $\log_{10}(n+h) - \log_{10}n : \log_{10}(n+h') - \log_{10}n :: h : h'.$

Cor. 1.—$\log_{10}(n+1) - \log_{10}n = \frac{\mu}{n}$ = tabular difference.

Cor. 2.—If the tabular difference be denoted by δ, we have

$$\log_{10}(n+h) - \log_{10}n = h\delta. \qquad (256)$$

Hence we have the following rule:—*$log_{10}(n+h)$ is found from $log_{10}n$ by adding to it the tabular difference multiplied by h.*

89a. If we put $\log_{10}(n+h) - \log_{10} n = \Delta$, and $\log_{10}(n+h') - \log_{10}{}^{n} = \Delta'$, the proportion (255) may be written $\Delta : \Delta' :: h : h'$. In proving this proposition, only small terms of the first order were used. We shall now examine the amount of the correction, by retaining small terms of the second order. The proposition is used for a twofold purpose, which we shall consider separately—

1°. Given $\Delta, \Delta', h,$ to find h'.

By hypothesis $\Delta = \log_{10}\left(1+\frac{h}{n}\right)$; $\therefore 10^{\Delta} = 1 + \frac{h}{n}$

Hence
$$1 + \frac{\Delta}{\mu} + \frac{\Delta^2}{2\mu^2} + \&c. = 1 + \frac{h}{n};$$

or, retaining only small terms of the second order,

$$\frac{\Delta}{\mu} + \frac{\Delta^2}{2\mu^2} = \frac{h}{n}.$$

Similarly,
$$\frac{\Delta'}{\mu} + \frac{\Delta'^2}{2\mu^2} = \frac{h'}{n}.$$

Now the value of h' given by the equation (255) is

$$\frac{\Delta' h}{\Delta} = n\Delta'\left(\frac{1}{\mu} + \frac{\Delta}{2\mu^2}\right).$$

Hence the difference
$$\frac{n\Delta'(\Delta - \Delta')}{2\mu^2}.$$

And since Δ' and $\Delta - \Delta'$ are each less than Δ, this difference is

$$< \frac{n\Delta^2}{2\mu^2}, \text{ and therefore } < \frac{h^2}{2n};$$

that is, < a third proportional to $2n$ and h.

2°. Given Δ, h, h', to find Δ'.

We have
$$\Delta = \log_{10}\left(1+\frac{h}{n}\right) = \mu\left(\frac{h}{n} + \frac{h^2}{2n^2}\right),$$

retaining only two terms,

$$\Delta' = \mu\left(\frac{h'}{n} - \frac{h'^2}{2n^2}\right), \ldots$$

and the value of Δ' given by (255)

$$= \frac{\Delta h'}{h} = \mu h'\left(\frac{1}{n} - \frac{h}{2n^2}\right).$$

Therefore the difference $= \dfrac{\mu h'(h-h')}{2n^2}$.

And since h', and $h - h'$ are each less than h, this difference is

$$< \frac{\mu h^2}{2n^2}.$$

90. The following Forms show how logarithms are arranged in Tables:—

I.—FORM FOR NUMBERS FROM 1 TO 1000.

No.	Logarithms.	No.	Logarithms.
2	·3010300	852	·9304396
3	·4771213	853	·9309490
4	·6020600	854	·9314579

II.—FORM FOR NUMBERS FROM 1000 TO 10,000.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
2583	4121244	1412	1580	1748	1917	2085	2253	2421	2589	2757	168
84	2925	3093	3261	3429	3597	3765	3933	4101	4269	4437	,,
85	4605	4773	4941	5109	5277	5445	5613	5781	5949	6117	,,
86	6285	6453	6621	6789	6957	7125	7293	7461	7629	7796	,,
Propl. Parts.		17	34	50	67	84	101	118	134	151	—

In Form I. all the mantissae are given in full, and it is only necessary to supply the characteristics. Thus the log of 852 is 2·9309346. In Form II., the first three figures of the mantissa being the same for all the numbers contained in it, are inserted only in the first column, and the rest are supplied according to the digit placed in vertical and horizontal rows; thus, for

25856, we have, corresponding to 5 in the *vertical*, and 6 in the *horizontal* row, the number 5613, which with 412, which is common, gives ·4125613 for the mantissa of the log of 25856; and since the number has 5 figures to the left of the decimal point, the characteristic is 4. Hence log 25856 = 4·4125613. The numbers in the last row, called proportional parts, are calculated as follows:—If we take the difference between the logarithms of any two consecutive numbers in Form II. we get 168; and multiply by ·1, ·2, ·3, &c. (see § 89, *Cor.* 2), we get the proportional parts 17, 34, 50, &c. The use of these is to find the logarithms of numbers having 6 or more places of figures. Thus, to find log 258468:—

Now	log 25846 is 4·4123933;
therefore	log 258460 is 5·4123933.
In like manner,	log 258470 is 5·4124101.

Now the difference between 258460 and 258470 is 10, and the difference of the logarithms is ·0000168; also the difference between 258460 and 258468 is 8. Hence (§ 89) the corresponding difference of logarithms is the fourth proportional to 10, 8, and ·0000168; that is, ·0000134, which added to the logarithm of 258460 gives 5·4124067 as the logarithm of 258468. It is easy to see how the Table of proportional parts enables us to abridge this calculation.

EXERCISES.—XVI.

1. Given log 2 = ·3010300; find log 128, log 5·12, log ·0032.
2. ,, log 3 = ·4771213; ,, log 81, log 21·87, log ·243.
3. ,, log 7 = ·8450980; ,, log 2401, log 3·43, log ·16807.

The following examples show how to manage negative characteristics.

4. Find the product of ·00625 and ·07854.

We have $\log \cdot 00625 = \bar{3}\cdot 7958800$

$\log \cdot 07854 = \bar{2}\cdot 8950909.$

Hence $\log (\cdot 00625 \times \cdot 07854 = \bar{4}\cdot 6909709;$

$\therefore \quad \cdot 00625 \times \cdot 07854 = \cdot 000490875.$

In this example, after adding the mantissa, we have 1 to carry, which, added to $\bar{5}$, the sum of the negative characteristics $\bar{3}$ and $\bar{2}$, gives $\bar{4}$.

5. Find the quotient of ·012345 by 54·86.

We have $\log \cdot 012345 = \bar{2}\cdot 0914911$

$\log 54\cdot 86 = 1\cdot 7393824.$

Hence, $\log \text{quotient} = \bar{4}\cdot 3520087;$

$\therefore \quad \text{quotient} = \cdot 00022491.$

In the subtraction of the mantissa we have 1 to carry, which, added to 1, the characteristic in the lower line, makes 2; and this, taken from $\bar{2}$, leaves $\bar{4}$.

6. Required the square root of ·00366326.

$\log \cdot 00366326 = \bar{3}\cdot 5638677.$

Hence, $\log \text{square root} = \bar{2}\cdot 7819338;$

$\therefore \quad \text{square root} = \cdot 0744618.$

In dividing $\bar{3}\cdot 5638677$ by 2, we put $\bar{4}$ instead of $\bar{3}$; and, to balance this we put 1 before the first figure in the mantissa, making it 15, and then divide.

7. Calculate the logarithms of the following numbers, making use of those given in Examples 1, 2, 3:—

1°. ·0020736 4·32 98 6·86 17·28 ·00336.

2°. $\cdot 042\frac{1}{2}$ $\cdot 686\frac{1}{4}$ $\cdot 063\frac{1}{3}$ $392\frac{1}{6}$ $882\frac{1}{10}$ $1\cdot 701\frac{1}{2}$.

8. Find how many digits in 3^{64}.

9. Prove $$\log_e 2 = \frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \frac{1}{5\,.\,6} \dots$$

10. ,, $$1 - \log_e 2 = \frac{1}{2\,.\,3} + \frac{1}{4\,.\,5} + \frac{1}{6\,.\,7} \dots$$

11. ,, $$\log_e 4 - 1 = \frac{2}{1\,.\,2\,.\,3} + \frac{2}{3\,.\,4\,.\,5} + \frac{2}{5\,.\,6\,.\,7}, \text{ \&c.}$$

12. Prove $\log_{10}(1+x) = \mu\,\{x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \&c.\}$. (257)

13. ,, $\log_{10}(n+1) = \log_{10}(n) + 2\mu\left\{\dfrac{1}{2n+1} + \dfrac{1}{3}\left(\dfrac{1}{2n+1}\right)^3 + \&c.\right\}$. (258)

14. Prove that if x be whole or fractional, but not incommensurable, e^x is incommensurable. The proof is the same as § 80, *Cor.* 3.

15. Prove that e cannot be a root of a quadratic equation with rational coefficients.

For, if e be a root of $$px^2 + qx + r = 0,$$

we have $$pe^2 + qe + r = 0;$$

$$\therefore\ pe + q + re^{-1} = 0,$$

and substituting for e and e^{-1} the values

$$1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}}, \ \&c., \text{ and } 1 - \frac{1}{1} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}}, \ \&c.,$$

the proposition may be proved, as in § 80, *Cor.* 3.

16. Sum the series $\dfrac{1}{\underline{|4}} + \dfrac{2}{\underline{|6}} + \dfrac{4}{\underline{|8}}$, &c., to infinity.

17. ,, ,, $\dfrac{1^2}{\underline{|2}} + \dfrac{2^2}{\underline{|3}} + \dfrac{3^2}{\underline{|4}}$, &c., to infinity.

18. Given $\cot 2^x\alpha - \cot 2^{x+1}\alpha = \operatorname{cosec} m\alpha$, finl x.

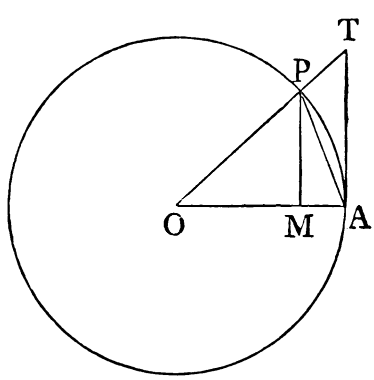
CHAPTER IV.

TRIGONOMETRIC TABLES.

SECTION I.—CONSTRUCTION OF TABLES OF CIRCULAR FUNCTIONS.

91. IN order that Trigonometry may be of practical use, it is necessary to possess Tables by means of which we can find the circular functions of an arc, when the arc is given, or, conversely, find the arc, when the arithmetical value of any of its circular functions is given. For this purpose it is sufficient to have Tables giving the functions of arcs increasing by small intervals from 0° to 45°; such as LALANDE's, which proceed by differences of $1'$, or CALLET's, by differences of $10''$. In fact— 1°. We shall see in the next section that by interpolation we can obtain the circular functions of other arcs between 0° and 45°. 2°. By means of complements we reduce arcs from 45° to 90°. 3°. That the circular functions of arcs in the 2nd, 3rd, and 4th, are the same, except in sign, as those in the first. We commence our calculations by seeking $\sin 1'$ and $\sin 10''$.

LEMMA I.—*If θ be the circular measure of an arc in the first quadrant, $\sin\theta$, θ, $\tan\theta$, are in ascending order of magnitude.*



DEM.—Let the arc AP of the unit circle be θ; then $MP = \sin\theta$, and $AT = \tan\theta$. Now the triangle OAP, the sector OAP, and the triangle OAT, are in ascending order of magnitude. Hence

$$\tfrac{1}{2}\,OA \, . \, MP < \tfrac{1}{2}\,OA \, . \, \text{arc } AP \;[\text{EUC. VI. xx., Ex. 14}] < \tfrac{1}{2}\,OA \, . \, AT;$$

$$\therefore\; MP < \text{arc } AP < AT,$$

which proves the proposition.

Lemma II.—If θ tend towards zero, we can substitute θ for $\sin\theta$.

Dem.—We have, lemma I., $\sin\theta < \theta, < \tan\theta$;

$$\therefore\quad 1 < \frac{\theta}{\sin\theta}, < \frac{1}{\cos\theta};$$

but the limit of $\cos\theta$ is 1 (§ 19). Hence, limit of

$$\frac{\theta}{\sin\theta} \text{ is } 1,$$

and therefore we can put $\dfrac{\theta}{\sin\theta} = 1 + \epsilon$

(ϵ being a quantity which tends towards zero, as θ approaches zero). Hence $\theta - \sin\theta = \epsilon\sin\theta$; that is, the difference $\theta - \sin\theta$ is a very small fraction of $\sin\theta$, when θ is very small. Hence, when θ diminishes indefinitely, we can for $\sin\theta$ substitute θ.

This lemma is important not only in Trigonometry, but in all the higher branches of Mathematics.

92. Calculation of sin 1′ and sin 10″.

It has been proved [Ex. VII. 50] that if θ be the circular measure of an angle, $\sin\theta$ is $> \theta - \dfrac{\theta^3}{6}$; but, Lemma I., $\sin\theta < \theta$. Hence $\sin\theta$ lies between θ and $\theta - \dfrac{\theta^3}{6}$. Now, if θ be the circular measure of $1'$, we have

$$\theta = \frac{2\pi}{360 \times 60} = \frac{3{\cdot}141592653589}{10800};$$

or $\theta = {\cdot}00029088820866$, correct to 14 decimal places.

Hence $\dfrac{\theta^3}{6} = {\cdot}00000000000401$, ,, ,, ,,

$\therefore\quad \theta - \dfrac{\theta^3}{6} = {\cdot}0002908820465$, ,, ,, ,,

Hence the values of θ and $\theta - \frac{\theta^3}{6}$, when θ denotes $1'$, agree as far as the first eleven decimal places; and since $\sin\theta$ lies between θ and $\theta - \frac{\theta^3}{6}$, we infer that

$$\sin 1' = \cdot 00029088820. \qquad (259)$$

Correct as far as the first eleven decimal places.

Cor. 1.—*The sines of all angles less than* $1'$ *and their circular measures agree as far as the first eleven decimal places.*

Cor. 2.—If n denote any number of seconds less than 60, then, approximately,

$$\sin n'' = n \sin 1''. \qquad (260)$$

Dem.—The sine of n'' is approximately equal to the circular measure of n'' (*Cor.* 1), and $\sin 1''$ to the circular measure of $1''$; but circular measure of n'' is equal to n times the circular measure of $1''$. Hence $\sin n'' = n \sin 1''$.

Cor. 3.—In the same manner,

$$\sin 10'' = \cdot 000048481368. \qquad (261)$$

Correct as far as twelve decimal places.

Cor. 4.—If $2(1 - \cos 10'') = k$; then

$$k = \cdot 0000000023504. \qquad (262)$$

Correct to thirteen decimal places.

93. Def.—*A series* $S \equiv u_1 + u_2 + u_3 + \ldots u_n$, *which is such that any term is the sum of a certain number of the preceding terms, multiplied respectively by given constants (called* Constants of Relation), *is called a* recurring *series.* Thus—2, 4, 14, 46, 152, &c., is a recurring series; for any term is equal to the sum of the two preceding ones, multiplied by the constants 1, 3, respectively.

94. *The sines of a series of angles in AP form a recurring series, whose constants of relation are* -1, *and twice the cosine of the common difference.*

DEM.—Let α, $\alpha + \beta$, $\alpha + 2\beta$ . . . be the angles; then, taking any three consecutive terms, such as $\alpha + \beta$, $\alpha + 2\beta$, $\alpha + 3\beta$, we see that

$$\sin(\alpha + 3\beta) = -\sin(\alpha + \beta) + 2\sin(\alpha + 2\beta)\cos\beta.$$

Hence the proposition is proved.

In like manner, the cosines of a series of angles in AP form a recurring series, whose constants are -1, *and twice the cosine of the common difference.*

95. To Construct a Table of Sines and Cosines.

Suppose the arcs to increase by $1'$; then, since

$$\sin 1' = \cdot 00029088020,$$

we get
$$\cos 1' = \cdot 99999995769;$$

but
$$\sin 2' = 2\sin 1'\cos 1'; \quad \therefore \sin 2' \text{ is known.}$$

Hence, by § 94, we have—

$$\sin 3' = 2\sin 2'\cos 1' - \sin 1',$$

$$\sin 4' = 2\sin 3'\cos 1' - \sin 2',$$

$$\sin 5' = 2\sin 4'\cos 1' - \sin 3'.$$

. . . . . .

Again, we have—

$$\cos 2' = 2\cos^2 1' - 1,$$

$$\cos 3' = 2\cos 2'\cos 1' - \cos 1',$$

$$\cos 4' = 2\cos 3'\cos 1' - \cos 2'.$$

. . . . . .

We can proceed in this manner until we come to the sine and the cosine of 45°; and since the sine of an angle is equal to the cosine of its complement, we have virtually calculated the sines and the cosines from 0° to 90°.

The method of this § is due to THOMAS SIMPSON. (See LECOINTE's *Trigonometry*, page 65.)

96. Another Method.—If in the identity

$$\sin(n+1)\alpha = 2\sin n\alpha\cos\alpha - \sin(n-1)\alpha$$

we put
$$2(1-\cos\alpha) = k,$$

we get
$$\sin(n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin(n-1)\alpha - k\sin n\alpha. \tag{263}$$

This formula enables us to construct a Table of sines of angles whose common difference is α. Thus, if we wish to construct Tables such as CALLET's, which proceed by common differences of $10''$, we put $\alpha = 10''$, and $n = 1, 2, 3,$ &c.

Thus
$$\sin 20'' - \sin 10'' = \sin 10'' - k\sin 10'',$$

$$\sin 30'' - \sin 20'' = \sin 20'' - \sin 10'' - k\sin 20'',$$

$$\sin 40'' - \sin 30'' = \sin 30'' - \sin 20'' - k\sin 30'', \text{ \&c.}$$

These equations give in succession $\sin 20''$, $\sin 30''$, $\sin 40''$, &c. It is seen that the most laborious part of the work is multiplying by the value of k, the smallness of which (equation 262) facilitates the process. When we have in this manner computed the sines of angles up to $3°$, we can verify our result by comparing the value obtained with that furnished by equation (124).

In the same manner a Table of COSINES can be constructed by means of the formula—

$$\cos n\alpha - \cos(n+1)\alpha = \cos(n-1)\alpha - \cos n\alpha + k\cos n\alpha. \tag{264}$$

Thus we get

$$\cos 10'' - \cos 20'' = 1 - \cos 10'' + k\cos 10'',$$

$$\cos 20'' - \cos 30'' = \cos 10'' - \cos 20'' + k\cos 20'',$$

$$\cos 30'' - \cos 40'' = \cos 20'' - \cos 30'' + k\cos 30'', \text{ \&c.}$$

Thus we have in succession the cosines of $20''$, $30''$, $40''$, &c.; and we can verify, as in the case of the sines, by equation (125).

97. The sines and cosines of angles from 3° to 30° may be calculated by the formulae—

$$\sin(\alpha+\beta)=\frac{(\sin\alpha+\sin\beta)(\sin\alpha-\sin\beta)}{\sin(\alpha-\beta)}, \tag{265}$$

$$\cos(\alpha+\beta)=\frac{(\cos\alpha+\sin\beta)(\cos\alpha-\sin\beta)}{\cos(\alpha-\beta)}. \tag{266}$$

Thus
$$\sin 4^\circ=\frac{(\sin 3^\circ+\sin 1^\circ)(\sin 3^\circ-\sin 1^\circ)}{\sin 2^\circ}, \tag{267}$$

$$\cos 4^\circ=\frac{(\cos 3^\circ+\sin 1^\circ)(\cos 3^\circ-\sin 1^\circ)}{\sin 2^\circ}. \tag{268}$$

98. The sines and cosines from 30° to 45° may be calculated by the formulae—

$$\sin(30+\alpha)=\cos\alpha-\sin(30-\alpha), \tag{269}$$

$$\cos(30+\alpha)=\cos(30-\alpha)-\sin\alpha. \tag{270}$$

Thus, if we put $\alpha=10'',\ 20'',$ &c., we shall have

$$\sin 30^\circ\ 0'\ 10''=\cos 10''-\sin 29^\circ\ 59'\ 50'',$$

$$\cos 30^\circ\ 0'\ 10''=29^\circ\ 59'\ 50''-\sin 10''.$$

And we can compute also Tables of sines and cosines up to 45°, and therefore to 90°.

99. The following formulae of verification are used for testing the calculation:—

1°. $$\sin(45^\circ+\alpha)=\sin(45^\circ-\alpha)+\sqrt{2}\sin\alpha. \tag{271}$$

2°. $$\sin(60^\circ+\alpha)=\sin(60^\circ-\alpha)+\sin\alpha. \tag{272}$$

3°. Euler's formulae (192). 4°. Legendre's formulae (193), (194).

Euler's may be written as follows:—

$$\sin\alpha=\cos(54^\circ+\alpha)-\cos(54^\circ+\alpha)+\cos(18^\circ-\alpha)-\cos(18^\circ+\alpha). \tag{273}$$

100. Tables of tangents and cotangents are calculated by the formulae—

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \quad \cot\alpha = \frac{1}{\tan\alpha}.$$

These functions having been calculated up to 45°, those over 45° are got from the formulae

$$\tan(45° + \alpha) = \cot(45° - \alpha),$$

$$\cot(45° + \alpha) = \tan(45° - \alpha),$$

and the identity

$$\tan(45° + \alpha) = \tan(45° - \alpha) + 2\tan 2\alpha. \qquad (274)$$

Exercises.—XVII.

1. Prove Delambre's formulae—

$$\sin(\alpha + 1°) - \sin\alpha = \sin\alpha - \sin(\alpha - 1°) - 4\sin^2\alpha\sin^2 30', \qquad (275)$$

$$\sin(\alpha + 1') - \sin\alpha = \sin\alpha - \sin(\alpha - 1') - 4\sin^2\alpha\sin^2 30''. \qquad (276)$$

2. Prove $\quad \sin 9° + \sin 27° + \sin 81° = \sin 45° + \sin 63°.$

3. ,, $\quad \sin 10° + \sin 26° + \sin 82° = \sin 46° + \sin 62°.$

4. Prove that the inscription of a regular heptagon in a circle depends on the solution of the cubic

$$x^3 - x^2 - 2x + 1 = 0. \qquad (277)$$

Dem.—Put

$$\alpha = \frac{\pi}{7}, \quad \text{then} \quad \cos 4\alpha + \cos 3\alpha = 0,$$

or $$4\cos^3\alpha - 3\cos\alpha + 1 - 8\cos^2\alpha + 8\cos^4\alpha = 0.$$

Change signs, divide by $\cos\alpha - 1$, and put $2\cos\alpha = x$, and we get the required equation.

5. Given $\quad \cos^3 2\theta = \cos^4\theta + 3\sin^4\theta$, find θ.

6. Prove that $2\sin 6°$ is a root of the equation

$$x^5 - 5x^3 + 5x - 1 = 0. \qquad (278)$$

7. Given

$\cos\theta + \sin 3\theta + \cos 5\theta + \sin 7\theta \ldots \sin(4n - 1)\theta = \frac{1}{4}(\sec\theta + \operatorname{cosec}\theta)$, find θ.

8. If $\alpha, \beta, \gamma, \delta$, be four solutions of the equation

$$\tan(\theta + 45^\circ) = 3\tan 3\theta,$$

no two of which have equal tangents, prove that

$$\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = 0, \quad \tan 2\alpha + \tan 2\beta + \tan 2\gamma + \tan 2\delta = 0.$$

9. If $\alpha = \frac{\pi}{17}$, prove that $2\cos\alpha$ is a root of the equation

$$x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 = 0. \qquad (279)$$

By hypothesis, $\cos 9\alpha + \cos 8\alpha = 0$,

and then expressing $\cos 9\alpha$ and $\cos 8\alpha$ in terms of $\cos\alpha$, and proceeding as in Ex. 4.

Section II.—Interpolation.

101. In Lalande's Tables, the circular functions are given for all the angles in the first quadrant calculated for degrees and minutes; and in those of Callet, of Dupuis, of Schrön, for angles differing only by $10''$. When, therefore, an angle, besides degrees and minutes, contains a number of seconds which is not a multiple of 10, its sine, cosine, &c., cannot be found in the Tables. In such cases we deduce from those of the angle in the Table which is nearest to the angle whose function is sought by means of the following principle:—*When the increments of an angle are small, the increments of its circular functions are approximately proportional to the increments of the angle.*

This is usually admitted as a result of the inspection of the Tables. We shall show by a diagram the amount of the error for the sine, the cosine, and the tangent, and shall find that in all cases it is very small for the sine and the cosine, but that for the tangent it may become very large.

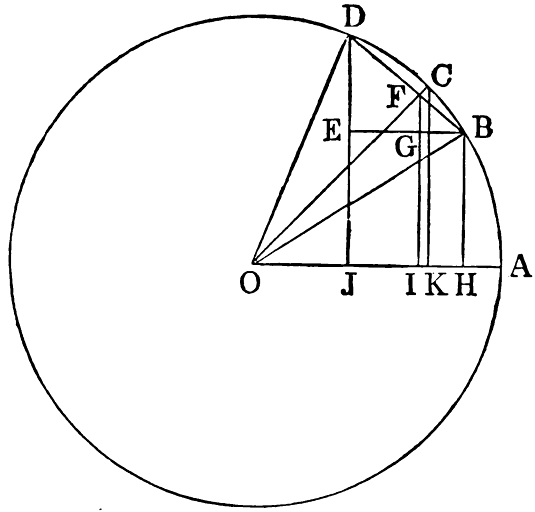
102. Let AB, AC, AD be three arcs on the unit circle. Let BD be $10''$, and BCh''. Join BD, cutting OC in F. Let O be the centre; draw BE parallel and FI perpendicular

to AO, cutting BE in G. Now, since OBD is an isosceles triangle,

$$BF : FD :: \sin BOF : \sin FOD :: \sin h'' : \sin (10 - h)'';$$

but $$\sin h'' : \sin (10 - h)'' :: h \sin 1'' : (10 - h) \sin 1''$$

(§ 92, *Cor.* 2);



$$\therefore \quad BF : FD :: h : 10 - h.$$

Hence $$BF : BD :: h : 10;$$

$$\therefore \quad DE : FG :: 10 : h.$$

Now if FI were the sine of the arc AC, FG would be the increment of sine corresponding to h'', and DE being the increment corresponding to $10''$, the proposition would be true. Hence it follows that the absolute error is equal to $KC - IF$, and the relative error is the ratio of this to CK. Therefore the relative error is the ratio of $CF : CO$. Hence the relative error is a maximum when BD is bisected. In this case it is ·000000002938. If BD were $1'$, the greatest relative error would be

·0000000105468.

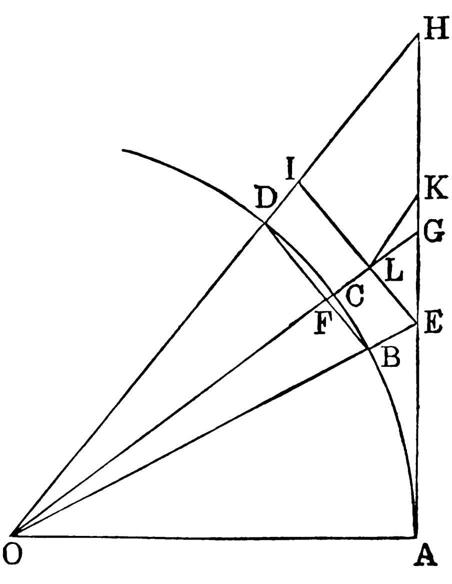
Cor. 1.—The relative error is independent of the arc AB, and depends only on the magnitude of the arc BD, and the position of the point C in it.

Cor. 2.—If the three points B, C, D were collinear, the proposition would be strictly accurate.

The smallness of the deviation from a right line may be inferred from the fact that the chord of an arc of $1'$ on the equator is about 6000 feet, and at its middle point is only about $2\frac{1}{2}$ inches from the surface. The chord of an arc of $10''$ is 1000 feet, and at its middle point is only $\frac{1}{14}$ of an inch from the surface. The collinearity of the three points is assumed without being formally stated in SERRET's *Trigonometry*, 6th edition, page 72.

Cor. 3.—Since we may replace the arcs AB, AC, AD by their complements, we see that the proposition holds for the increments of the cosines.

103. Let the arcs be the same as in § 102; AE, AG, AH their tangents; then, if AE, AH be given, and we want to find AG, draw EI parallel to BD, cutting OG in L, and draw LK parallel to OH. Since, by similar triangles,



$$IE : LE :: HE : KE,$$

we have

$$BD : BF \text{ or } 10'' : h'' :: HE : KE,$$

and

$$HE = \tan(\theta + 10'') - \tan\theta.$$

Hence the proportion

$$\tan(\theta + 10'') - \tan\theta : \tan(\theta + h'') - \tan\theta :: 10'' : h''$$

gives

$$KE = \tan(\theta + h'') - \tan\theta; \text{ that is } = AG - AE.$$

Therefore KG represents the absolute error, and $KG \div AG$ is the relative error. From similar triangles we get

$$OG : GH :: LG : GK; \quad \therefore \quad GK = \frac{GH \, . \, LG}{OG}$$

$$= \frac{\{\tan(\theta + 10'') - \tan(\theta + h'')\}\{\sec(\theta + h'') - \sec\theta\}}{\sec(\theta + h'')}$$

$$\frac{2 \sin(10'' - h'') \, . \sin(\theta + \tfrac{1}{2}h'') \sin \tfrac{1}{2}h''}{\cos(\theta + 10'') \cos\theta \, . \cos(\theta + h'')}$$

$$= \frac{(10 - h)\, h \sin^2 1'' \, . \sin(\theta + \tfrac{1}{2}h)}{\cos(\theta + 10'') \cos\theta \, . \cos(\theta + h'')}.$$

Hence

$$GK \div AG = \frac{(10 - h) h \sin^2 1'' . \sin(\theta + \tfrac{1}{2}h'')}{\cos\theta \, . \cos(\theta + 10'') \, . \sin(\theta + h'')} = \frac{(10 - h)\, h \sin^2 1''}{\cos\theta \, . \cos(\theta + 10'')}$$

nearly, when θ is large.

The maximum value of the numerator is when $h = 5$. Hence the maximum value of the relative error $= (5 \sin 1'' \sec \theta)^2$. Therefore for values of θ not greater than 80°, for which $\sec\theta$ is < 6, the maximum value is not greater than $(\cdot 000145444)^2$, or the maximum value of the relative error is $< \cdot 000000022$, and the absolute error is $< \cdot 000000132$.

Observation.—The results of the investigations in §§ 102, 103 may be verified by inspection of Tables, and similarly for the other circular functions.

104. Before giving applications of our rules, it is necessary to explain the manner in which the circular functions are arranged in Tables.

III.—Form of Tables for Natural Sines, Etc.

39°

′	Sine.	Diff.	Cosecant.	Diff.	Tangent.	Diff.	Cotangent.	Diff.	Secant.	Diff.	Cosine.	Diff.	′
41	·6385440		1·5560628		·8596297		1·2052190		1·2994011		·7695853		19
		2238		5487		5060		7132		3137		1857	
42	·6387678		1·5655141		·8601357		1·2045058		1·2997148		·7693996		18
		2238		5483		5062		7126		3140		1859	
43	·6389916		1·5649658		·8606419		1·2037932		1·3000288		·7692137		17
		2237		5477		5065		7122		3143		1859	
44	·6392153		1·5644181		·8611484		1·2030810		1·3003431		·7690278		16
		2237		5473		5067		7117		3145		1860	
45	·6394390		1·5638708		·8616551		1·2023693		1·3006576		·7688418		15
		2236		5467		5070		7142		3148		1860	
′	Cosine.	Diff.	Secant.	Diff.	Cotangent.	Diff.	Tangent.	Diff.	Cosecant.	Diff.	Sine.	Diff.	′

50°

The following remarks on the foregoing are important:—

1°. Degrees are marked both at the top and bottom, and minutes both in the first column at the *left*, and in the last column at the right-hand side. In seeking for a circular function of an angle, if it be less than 45°, the degrees are found at the top, and the minutes at the left-hand side. If greater than 45°, the degrees are found at the foot, and the minutes at the right-hand side. Thus the natural sine of 39° 44′ is ·6392153, and the natural tangent of 50° 18′ is 1·2045058.

2°. Every number in each column except those marked *Diff.* stands for two circular functions, the designations of which are given at the top and bottom of the column. Thus the number ·6385440 is sin 39° 41′, and also cos 50° 19′, the reason of which is obvious.

3°. The functions cosec, cotan, cosine, decrease as the angle increases.

4°. The numbers in the *Diff.* columns, called "Tabular Differences," are the differences between the consecutive numbers in the preceding columns. Thus 2238, the first number in the first diff. column, is the difference between the natural sines of 39° 41′ and 39° 42′, so that each is the difference between the circular functions of two angles which differ by 60″. The use of the diff. column is to find the functions of angles not given in the Tables.

Example.—Find the natural sine of 39° 41′ 25″.

The difference between 39° 41′ 25″ and 39° 41′ is 25″. And since the tabular difference 2238 corresponds to a difference of 60″, we have (§ 101) 60″ : 25″ : : 2238 : a fourth proportional which is the difference between sin 39° 41′ 25″ and sin 39° 41″. Hence, adding the fourth proportional 932 to ·6385440, we get ·6386372, which is the sine required.

105. Logarithms of Circular Functions.

In general,

$$\log_{10}\sin(\theta+h)-\log_{10}\sin\theta : \log_{10}\sin(\theta+h')-\log_{10}\sin\theta :: h : h'. \tag{280}$$

Dem.—Denote the arcs AB, AC, AD (§ 102) by θ, $\theta+h'$, $\theta+h$, respectively; then we have

$$\log_{10}\sin(\theta+h')-\log_{10}\sin\theta : \log_{10}\sin(\theta+h)-\log_{10}\sin\theta$$
$$:: \log_{10} CK-\log_{10} BH : \log_{10} DJ-\log_{10} BH;$$

but CK is equal to FI very nearly (§ 102). Hence

$$\log_{10}\sin(\theta+h')-\log_{10}\sin\theta :: \log_{10}\sin(\theta+h)-\log_{10}\sin\theta$$
$$:: \log_{10} FI-\log_{10} BH : \log_{10} DJ-\log_{10} BH.$$

Now, if the difference between DJ and BH be very small when compared to BH, we have (§ 89),

$$\log_{10} FI-\log_{10} BH : \log_{10} DJ-\log_{10} BH :: FG : DE;$$

that is, $\qquad :: BF : BD :: h' : h.$

Hence the proposition is proved.

There are two sources of approximation in this proposition—1°. We have substituted FI for CK; but when BD does not exceed 10″, this is so small that it may be disregarded (§ 102). 2°. If AB be not large when compared with BD, the proportion $\log_{10} FI-\log_{10} BH : \log_{10} DJ-\log_{10} BH :: FG : DE$ cannot be used (§ 89). Hence the proportion of this § will not hold with sufficient approximation to be employed in practice.

We shall now indicate the limits within which it cannot be used. Putting $BH=n$, $JD=n+h_1$, $IF=n+h_1'$, the proportion

$$\log_{10} JD-\log_{10} BH : \log_{10} FI-\log_{10} BH :: DE : FG$$

may be written $\qquad \Delta : \Delta' :: h_1 : h_1'.$

Now (§ 89*a*), the error in obtaining h_1' by this proportion is

$$< \frac{h_1^2}{2n}.$$

But we have seen (§ 102) that $h : h' :: h_1 : h_1'$. Hence, if

$$h_1 = kh, \quad h_1' = kh'.$$

Therefore the error in obtaining h' from the proportion

$$\Delta : \Delta' :: h : h',$$

that is from (280), is

$$< \frac{kh^2}{2n}, < \frac{h^2}{2n}, \text{ since } k < 1;$$

$$\text{but } \frac{h}{n} \text{ is easily seen to be } < \frac{\sin h}{\sin \theta}.$$

Hence the error is $$< \frac{h \sin h}{2 \sin \theta}.$$

Now, if h be $10''$ and $\theta = 1\frac{1}{2}°$, this error will be < the hundredth part of a second, and for larger values of θ it will be absolutely insensible. Again, if the proportion $\Delta : \Delta' :: h : h'$ be employed for obtaining Δ', it may be shown in a similar way from § 89, note, that the error is

$$< \frac{\mu}{2} \cdot \frac{\sin^2 h}{\sin^2 \theta}.$$

But if h be $10''$ and $\theta > 5°$, this will not amount to a unit in the seventh decimal place.

106. Since the sine of an angle, unless when it is right, is less than unity, the log sine will have a negative characteristic. This will be the case also with some of the other circular functions. In order to avoid this, the logarithm of each function was increased by 10 before being registered in the Tables. The logarithm so increased, called the *Tabular logarithm*, is usually denoted by the capital letter L. Thus: L sin 30° means the *Tabular logarithm* of sin 30°, and is greater by 10 than the *Natural logarithm* of sin 30°.

For example,

$$\sin 30^\circ = \cdot 5 \; ; \quad \therefore \; \log_{10} \sin 30^\circ = \bar{1}\cdot 6989700.$$

Hence
$$\text{L} \sin 30^\circ = 9\cdot 6989700.$$

It is now evident that the proportion in § 105 may be written

$$\text{L} \sin(\theta + h) - \text{L} \sin \theta : \text{L} \sin(\theta + h') - \text{L} \sin \theta :: h : h'. \quad (281)$$

107.—If in equation (281) we change

$$\theta = \frac{\pi}{2} - \theta',$$

we get, after omitting accents, and changing the signs of h, h',

$$\text{L} \cos(\theta + h) - \text{L} \cos \theta : \text{L} \cos(\theta + h') - \text{L} \cos \theta :: h : h'. \quad (282)$$

And from these two we get

$$\text{L} \tan(\theta + h) - \text{L} \tan \theta : \text{L} \tan(\theta + h') - \text{L} \tan \theta :: h : h'. \quad (283)$$

Observation.—The French and German Mathematicians are beginning to use the logarithms of the Natural sines, cosines, &c., instead of the Tabular Logarithms. Thus, Serret, Briot and Bouquet, Reidt, &c., subtract 10 from the index of the Tabular Logarithms before using it. Also in Dupuis's Tables the 10 has not been added. It is likely that English Mathematicians will soon follow their example.

IV.—Form for Logarithmic Sines, Etc.

39°

′	L sine.	Diff.	L cosecant.	L tangent.	Diff.	L cotangent.	L secant.	Diff.	L cosine.	′
41	9·8051908	1522	10·1948092	9·9189340	2571	10·0810660	10·1137432	1049	9·8862568	19
42	9·8053430	1521	10·1946570	9·9191191	2570	10·0808089	10·1138481	1049	9·8861519	18
43	9·8054951	1521	10·1945049	9·9194481	2570	10·0805519	10·1139530	1050	9·8860410	17
44	9·8056472	1519	10·1943528	9·9197051	2570	10·0802949	10·1140580	1050	9·8859420	16
′	L cosine.	Diff.	L secant.	L cotangent.	Diff.	L tangent.	L cosecant.	Diff.	L sine.	′

50°

If θ be any angle, $\sin\theta \,.\, \operatorname{cosec}\theta = 1.$

Hence $$\text{L}\sin\theta + \text{L}\operatorname{cosec}\theta = 20.$$

This explains why the sum of the tabular numbers in the columns L sine and L cosecant is in each case 20; when, therefore, the angle changes, one of these quantities will decrease by as much as the other increases. Hence we see why one column of differences does for both. Similar observations are applicable to the L tangent and L cotangent, the L secant and L cosine.

Exercises XVIII.

1. Given $$\text{L}\sin 37^\circ\, 43'\, 50'' = 9\cdot7867152,$$ $$\text{L}\sin 37^\circ\, 44'\, 40'' = 9\cdot7867424:$$

find $$\text{L}\sin 37^\circ\, 43'\, 56''.$$

If Δ, Δ' denote the log. differences for $10''$ and $6''$, respectively, we have

(§ 106) $$\Delta : \Delta' :: 10 : 6;$$

but $$\Delta = 272; \quad \therefore\ \Delta' = 163.$$

Hence $$\text{L}\sin 37^\circ\, 43'\, 56'' = 9\cdot7867315.$$

2. If the angle whose L sine is given be smaller than 5°, the tabular differences vary irregularly (§ 105), and the preceding method is inapplicable. The following solution is taken from Serret's *Trigonometry*:—

Let the arc whose L sine is given, when expressed in seconds, be denoted by a, and the arc whose L sine is required be $a + h$. Then we have

$$\sin(a+h) : \sin a :: a + h : a. \quad \text{(Comp. } \textit{Observ.}\text{, § 102.)}$$

Hence $$\text{L}\sin(a+h) = \text{L}\sin a + \log(a+h) - \log a. \tag{284}$$

Similarly, $$\text{L}\tan(a+h) = \text{L}\tan a + \log(a+h) - \log a. \tag{285}$$

Find $$\text{L}\sin 0^\circ\, 3'\, 27''\cdot355$$

Here we have $$a = 207, \quad h = \cdot355,$$

$$\text{L}\sin 207'' = 7\cdot0015451, \quad \log 207 = 2\cdot3159703,$$
$$\log 207\cdot355 = 2\cdot3167145.$$

Hence, substituting in (284), we get

$$\text{L}\sin 0^\circ\, 3'\, 27''\cdot355 = 7\cdot0022893.$$

Observation.—CALLET's Tables give the logarithms of the circular functions for every second of the first five degrees of the quadrant.

3. Given $L\cos 21° \, 32' \, 30'' = 9{\cdot}9685534$, $\Delta = \text{diff. for } 10'' = 84$,

find $$L\cos 21° \, 32' \, 34''{\cdot}7.$$

4. Given $L\tan 83° \, 7' \, 10'' = 10{\cdot}9184044$, $\Delta = \text{diff. for } 10'' = 1771$,

find $$L\tan 83° \, 7' \, 16''{\cdot}4.$$

5. Given $L\cot 68° \, 52' \, 50'' = 9{\cdot}5868773$, $\Delta = \text{diff. for } 10'' = 626$,

find $$L\cot 68° \, 52' \, 47''{\cdot}3.$$

In the following Exercises, 6 and 7, the logarithms of the circular functions are given to find the corresponding angles when they are not found in the Tables.

6. Find the angle whose L sine is 9·8835535, being given

$$L\sin 49° \, 53' \, 20'' = 9{\cdot}8835459, \quad L\sin 49° \, 53' \, 30'' = 9{\cdot}8835636.$$

Here $$\Delta = 177, \quad \Delta' = 76.$$

Hence the proportion $\Delta : \Delta' :: 10 : h$ gives $h = 4''{\cdot}29$;

therefore the required angle is $49° \, 53' \, 24''{\cdot}29$.

7. Given $$L\tan 79° \, 51' \, 40'' = 10{\cdot}7475657,$$ $$L\tan 79° \, 51' \, 50'' = 10{\cdot}7476872,$$

find the angle whose L tan is 10·7476532.

Here $$\Delta = 1215, \quad \Delta' = 875.$$

Hence $h = 7''{\cdot}2$; $\therefore$ the angle is $79° \, 51' \, 47''{\cdot}2$.

If the angle required be small, the proportion $\Delta : \Delta' :: 10 : h$ fails, and we must proceed as in Ex. 2. This will occur when the required angle is not greater than $1\frac{1}{2}°$.—SERRET's *Trig.*, page 76.

Observation.—It is impossible to determine an arc very near 90° by means of its L sine, nor an arc very near zero by means of its L cos. Thus, if the L cos be 9·9999991, the Tables show that this logarithm belongs to all arcs between 7′ 25″ and 6′ 45″. Thus there is an uncertainty of 40″.

Section III.—Transformation of Formulae into Logarithmic Form.

108. As calculations are principally performed by means of logarithms, it is necessary to know how to transform sums and differences into products. All the formulae of Chapter II., Section VI., are examples of this process. The following are further applications:—

1°. Transform $a\cos\theta \pm b\sin\theta$ into a product.

$$a\cos\theta \pm b\sin\theta = a(\cos\theta \pm \frac{b}{a}\sin\theta) = (a\cos\theta \pm \tan\phi\sin\theta),$$

if

$$\tan\phi = \frac{b}{a}.$$

Hence

$$a\cos\theta \pm b\sin\theta = \frac{a(\cos\theta \pm \phi)}{\cos\phi}. \qquad (286)$$

Similarly,

$$a\sin\theta \pm b\cos\theta = \frac{a\sin(\theta \pm \phi)}{\cos\phi}. \qquad (287)$$

2°. Transform $a \pm b$, $a \pm b \pm c$ into products.

It may occur that we may have occasion to find

$$\log(a \pm b), \quad \log(a \pm b \pm c),$$

where $\log a$, $\log b$, $\log c$ are more easily determined than a, b, c. For example, $\sin x = 0{\cdot}317 \cos 62° \, 17' \, 54'' + 0{\cdot}614 \tan 37° \, 35' \, 20''$, to find x.

$$a + b = a\left(1 + \frac{b}{a}\right) = a(1 + \tan^2\phi) = a\sec^2\phi,$$

if

$$\frac{b}{a} = \tan^2\phi;$$

$$a - b = a\left(1 - \frac{b}{a}\right) = a\cos^2\phi,$$

if

$$\frac{b}{a} = \sin^2\phi.$$

Hence $$\log(a+b) = \log a + 2\log\sec\phi, \tag{288}$$

$$\log(a-b) = \log a + 2\log\cos\phi. \tag{289}$$

To find $$\log(a \pm b \pm c),$$

we first find $\log(a \pm b)$, and then $\log(a \pm b \pm c)$.

109. To transform the roots of an equation of the 2nd degree.

1°. $x^2 - px = q, \quad x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q} = \frac{p}{2}\left(1 \pm \sqrt{1 + \frac{4q}{p^2}}\right);$

put

$$\left.\begin{aligned} \frac{4q}{p^2} = \tan^2\phi;\ \text{then } x = \frac{p}{2}\left(1 \pm \frac{1}{\cos\phi}\right) &= p\cos^2\tfrac{1}{2}\phi\sec\phi, \\ \text{or } &= -p\sin^2\tfrac{1}{2}\phi\sec\phi. \end{aligned}\right\} \tag{290}$$

2°. $x^2 - px + q = 0, \quad x = \frac{p}{2}\left(1 \pm \sqrt{1 - \frac{4q}{p^2}}\right) = \frac{p}{2}\left(1 \pm \sqrt{1 - \sin^2\phi}\right),$

if $$\sin^2\phi = \frac{4q}{p^2}.$$

Hence $$x = p\cos^2\tfrac{1}{2}\phi, \quad \text{or} \quad p\sin^2\tfrac{1}{2}\phi. \tag{291}$$

Exercises.—XIX.

1. Calculate x from the equation

$$\tan x = \tan 12°\ 40'\ 7'' + \tan 18°\ 39'\ 23''.$$

2. ,, $\tan x = -\sqrt{\frac{7}{15}}.$

3. ,, $\tan 3x = \sqrt{\frac{1 - \tan^2 27°\ 43'\ 17''}{1 + \tan^2 49°\ 18'\ 36''}}.$

4. ,, $14\tan^2 x - 39\tan x = 35.$

5. ,, $12\sin^2 x - \sin x = 6.$

Section IV.—Trigonometric Equations.

110. We have already resolved or proposed several trigonometric equations. The object of the following examples is to explain the methods to be adopted, in order that the solution may be performed by logarithms.

Example 1.— $$a \sin x + b \cos x = c. \tag{292}$$

Putting $\frac{b}{a} = \tan\phi$, we have $\sin(x+\phi) = \frac{c}{a}\cos\phi$.

In order that the problem may be possible, it is necessary to have $\frac{c^2}{a^2}\cos^2\phi \leqq 1$; that is, $c^2\cos^2\phi \leqq a^2$,

or
$$\frac{c^2}{1+\tan^2\phi} \leqq a^2;$$

therefore
$$c^2 \leqq a^2\left(1+\frac{b^2}{a^2}\right), \quad c^2 \leqq (a^2+b^2).$$

If $c^2 = a^2 + b^2$, we have $\sin(x+\phi) = \pm 1$;

$$\therefore \quad x + \phi = 2n\pi \pm \frac{\pi}{2}.$$

If $c^2 < a^2 + b^2$, and if α be an angle whose sine $= \frac{c}{a}\cos\phi$,

we have $x + \phi = 2n\pi + \alpha$, or $(2n+1)\pi - \alpha$.

Application, $5\cos x - 8\sin x = 7$; find x.

Example 2.— $$a\sin^2 x + 2b\sin x\cos x + c\cos^2 x = m. \tag{293}$$

Replacing m by $m(\sin^2 x + \cos^2 x)$, and dividing by $\cos^2 x$, we get

$$(a-m)\tan^2 x + 2b\tan x + (c-m) = 0,$$

or, say
$$\tan^2 x + 2\beta\tan x + \gamma = 0.$$

We solve this by the process of § 109; for

$$x^2 + px + q = 0.$$

Or thus:—The equation can be written

$$\frac{a(1-\cos 2x)}{2} + b \sin 2x + c\left(\frac{1+\cos 2x}{2}\right) = c,$$

which reduces to the preceding case.

EXAMPLE 3.— $\sin(a+x) = m\cos(b-x)$.

If we expand $\sin(a+x)$, $\cos(b-x)$,

and then divide by $\cos x$, we get

$$\tan x = \frac{m\cos b - \sin a}{\cos a + m \sin b}.$$

For a numerical application, it is better to write—

$$\frac{\sin(a-x)}{\sin\left(\frac{\pi}{2} - b + x\right)} = \frac{m}{1};$$

therefore
$$\frac{\sin(a-x) + \sin\left(\frac{\pi}{2} - b + x\right)}{\sin(a-x) - \sin\left(\frac{\pi}{2} - b + x\right)} = \frac{m+1}{m-1}.$$

Hence
$$\frac{\tan\left(\frac{\pi}{4} + \frac{a}{2} - \frac{b}{2}\right)}{\tan\left(\frac{a}{2} + \frac{b}{2} - \frac{\pi}{4} - x\right)} = \frac{m+1}{m-1}. \qquad (294)$$

EXAMPLE 4.— $x + y = a$, $\sin x - \sin y = b$.

From second equation,

$$2\sin\tfrac{1}{2}(x-y)\cos\tfrac{1}{2}(x+y) = b;$$

$$\therefore \quad \sin\tfrac{1}{2}(x-y) = \frac{b}{2\cos\frac{1}{2}a}.$$

Hence $x - y$ is found.

EXAMPLE 5.— $x - y = a, \quad \cos x \cos y = b.$

From second equation, we get

$$\cos(x+y) + \cos(x-y) = 2b.$$

Hence $\quad \cos(x+y) = 2b - \cos a,$ which gives $x + y$.

EXAMPLE 6.— $x + y = a, \quad \tan x \cot y = b.$

Here we have $$\frac{\sin x \cos y}{\cos x \sin y} = \frac{b}{1}.$$

therefore $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{b+1}{b-1}$; that is, $\dfrac{\sin a}{\sin(x-y)} = \dfrac{b+1}{b-1}$.

Hence $x - y$ is found.

EXERCISES XX. ON CHAPTER IV.

1. Given $\tan\left(\frac{\pi}{4}+\theta\right) + \tan\left(\frac{\pi}{4}-\theta\right) = \left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}}$, find $\cos 2\theta$.

2. ,, $\sec^2 x + \sec 2x = 8 - \frac{10}{\sqrt{3}}$, prove $x = n\pi \pm \frac{\pi}{12}$.

3. ,, $(2\cos\alpha - 1)(2\cos 2\alpha - 1)(2\cos 4\alpha - 1)\ldots(2\cos 2^{n-1}\alpha - 1)$

$$= \frac{2\cos 2^n\alpha + 1}{2\cos\alpha + 1}. \qquad (295)$$

4-10. Find x from the following equations:—

4. $a\tan x + b\cot x = c.$

5. $\sin^4 x + \cos^4 x = \frac{5}{8}.$

6. $\sin 3x = 2\sin^2 x.$

7. $\tan^2 x = 4\sin x.$

8. $\tan x = 4\sin\frac{x}{2}.$

9. $\sin x + \cos x = \frac{4}{7}.$

10. $27\cos x + 50\sin x = 36{\cdot}97.$

11–16. Find x, y from the following simultaneous equations:—

11. $$x - y = a, \quad \sin x + \cos y = b.$$

12. $$x - y = a, \quad \sin x \cos y = b.$$

13. $$x - y = a, \quad \tan x + \tan y = b.$$

14. $$x + y = a, \quad \sec x + \sec y - b.$$

15. $$x - y = a, \quad \operatorname{cosec} x - \operatorname{cosec} y = b.$$

16. $$x + y = 52^\circ\ 47'\ 3'', \quad \sin x + \sin y = \tfrac{6}{7}.$$

17. If θ be the circular measure of n'',

prove $$\log n + \log \frac{\sin\theta}{\theta} = \text{L} \sin n'' - \text{L} \sin 1''. \quad \text{(Delambre.)}$$

Dem.—When n is small, we have approximately $\theta = n \sin 1''$;

$$\therefore \quad \frac{\sin\theta}{\theta} = \frac{\sin n''}{n \sin 1''};$$

$$\therefore \quad \log n + \log \frac{\sin\theta}{\theta} = \text{L} \sin n'' - \text{L} \sin 1''. \tag{296}$$

18. When θ is very small,

$$\log \sin\theta = \log\theta + \tfrac{1}{3} \log \cos\theta. \quad \text{(Maskelyne.)} \tag{297}$$

19. In the same case,

$$\log \tan\theta = \log\theta - \tfrac{2}{3} \log \cos\theta. \quad (\textit{Ibid.}) \tag{298}$$

20. Given $\tan^2(\alpha + x) + \tan^2(\alpha - x) = \tan\alpha \tan x$;

prove $$(\cos x \pm \cos\alpha)^2 = 1.$$

21. Prove

$$\tan(\theta + h) - \tan\theta = \tan h \sec^2\theta\,(1 + \tan h \tan\theta + \tan^2 h \tan^2\theta + \&c.)$$

22. Given $\sec\alpha \sec\theta + \tan\alpha \tan\theta = \sec\beta$, find $\tan\theta$.

23. ,, $\dfrac{\cos x}{a} = \dfrac{\cos 2x}{b} = \dfrac{\cos 3x}{c}$, prove that $\sin^2\dfrac{x}{2} = \dfrac{2b - (a + c)}{4b}$.

24. ,, $\dfrac{\sin x}{a} = \dfrac{\sin 3x}{b} = \dfrac{\sin 5x}{c}$, prove that $a(a + b + c) = b^2$.

25. ,, $(\tan 2x + a - 1)/(\tan 2x + a + 1) = \tan x$, find $\tan x$.

26. Given $4(1 + 8 \operatorname{cosec}^2 4\theta) = (\tan^2\theta + \cot^2\theta)(\tan^2\alpha + \cot^2\alpha)$, find $\cos 2\theta$.

27. ,, $(\tan^2\theta + \tan^2\phi)/(\tan^2\theta - \tan^2\phi) = \frac{5}{3}$, $\sin(\theta - \phi) = \frac{1}{10}$;

find θ and ϕ.

28. ,, $\tan(x - y) = \sin(x + y) = \frac{1}{2}(\tan x - \tan y)$, find x, y.

29. ,, $x \cos\alpha + x' \sin\alpha = x \cos\beta + x' \sin\beta = y \cos\gamma + y' \sin\gamma$

$$= y \cos\delta + y' \sin\delta = 1;$$

prove $$(x + y)/(x - y) = 2(1 - s_4)/(s_1 + s_3),$$

where s_n denotes the sum of the products of

$$\tan\frac{\alpha}{2},\quad \tan\frac{\beta}{2},\quad \tan\frac{\gamma}{2},\quad \tan\frac{\delta}{2},\ \text{taken } n \text{ by } n.$$

30. Prove that when x is less than unity, the sum of the infinite sines,

$$\cos\alpha + x\cos 2\alpha + x^2\cos 3\alpha + \&c., \text{ is } \frac{\cos\alpha - x}{1 - 2x\cos\alpha + x^2}. \qquad (299)$$

Dem.—In the identity

$$\cos(n + 1)\alpha \, . \, x^n = 2\cos n\alpha \, . \cos\alpha \, . \, x^n - \cos(n - 1)\alpha \, . \, x^n,$$

put $n = 0, 1, 2, 3,$ &c. Add, and we get

$$\cos\alpha + x\cos 2\alpha + x^2\cos 3\alpha + \&c. = \cos\alpha - x$$
$$+ (2x\cos\alpha - x^2)(\cos\alpha + x\cos 2\alpha + x^2\cos 3\alpha + \&c.)$$

Hence, by transposition and division, the proposition is proved.

31. Prove

$$1 + 2x\cos\alpha + 2x^2\cos 2\alpha + 2x^3\cos 3\alpha + \&c. = \frac{1 - x^2}{1 - 2x\cos\alpha + x^2}. \qquad (300)$$

32. If $\alpha + \beta + \gamma = \pi$ and $\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$;

prove that

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma \text{ and } \operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma. \qquad (301)$$

33. Prove $\log\cot\theta = \cos 2\theta + \dfrac{\cos^3 2\theta}{3} + \dfrac{\cos^5 2\theta}{5} + \&c.$ (302)

34. Prove $2 \cos^3\theta\,(\cos\theta - \cos\alpha) - 2 \sin^3 (\sin\theta - \sin\alpha) - \cos^2\theta + \cos^2\alpha$

$$= 4 \sin^3 \frac{\theta - \alpha}{2} \sin \frac{3\theta + \alpha}{2}.$$

35. Given $\cos\alpha \tan^2\alpha\,(\cos x - \cos\beta) = \cos\beta \tan^2\beta\,(\cos x - \cos\alpha)$;

prove $$\tan^2\frac{x}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}.$$

36. If the value of the fraction

$$(a\cos(\theta + \alpha) + b\sin\theta)/(a'\sin(\theta + \alpha) + b'\cos\theta)$$

be independent of θ, prove that $\sin\alpha = \dfrac{aa' - bb'}{a'b - ab'}$.

(First put $\theta = 0$ and then $= -\alpha$, and equate results.)

37–43. Prove the following identities:—

37. $\sin^2 x \sin 3x \sin 7x + \sin^2 5x \sin x \sin 3x = \sin^2 2x \sin 4x \sin 6x.$

38. $\cos^2 x \cos 3x \cos 7x + \sin^2 2x \sin 4x \sin 6x = \cos^2 5x \cos x \cos 3x.$

39. $\sin^2 x \sin 2x \sin 6x + \sin^2 4x \sin x \sin 3x = \sin^2 2x \sin 3x \sin 5x.$

40. $\cos^2 x \cos 2x \cos 6x + \sin^2 2x \sin 3x \sin 5x = \cos^2 4x \cos x \cos 3x.$

41.
$$\begin{aligned}
&\sin\alpha \,.\, \sin\beta \,.\, \sin(\alpha - \beta)\{\sin^2\alpha + \sin^2\beta + \sin^2(\alpha - \beta)\}\\
&\quad + \sin\beta \,.\, \sin\gamma \,.\, \sin(\beta - \gamma)\{\sin^2\beta + \sin^2\gamma + \sin^2(\beta - \gamma)\}\\
&\quad + \sin\gamma \sin\alpha \,.\, \sin(\gamma - \alpha)\{\sin^2\gamma + \sin^2\alpha + \sin^2(\gamma - \alpha)\}\\
&\quad + \sin(\alpha - \beta)\sin(\beta - \gamma)\sin(\gamma - \alpha)\\
&\qquad \{\sin^2(\alpha - \beta) + \sin^2(\beta - \gamma) + \sin^2(\gamma - \alpha)\} = 0.
\end{aligned}$$

42.
$$\begin{aligned}
&\sin(\alpha + \beta)\sin(\alpha - \beta)\sin(\gamma + \delta)\sin(\gamma - \delta)\\
&\quad + \sin(\beta + \gamma)\sin(\beta - \gamma)\sin(\alpha + \delta)\sin(\alpha - \delta)\\
&\quad + \sin(\gamma + \alpha)\sin(\gamma - \alpha)\sin(\beta + \delta)\sin(\beta - \delta) = 0.
\end{aligned}$$

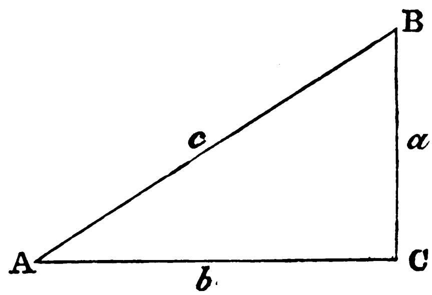
43.
$$\begin{aligned}
&\sin(\alpha + \beta)\sin(\alpha - \beta)\sin(\gamma + \delta)\sin(\gamma - \delta)\\
&\quad + \cos(\alpha + \beta)\cos(\alpha - \beta)\cos(\gamma + \delta)\cos(\gamma - \delta)\\
&\quad = \sin(\gamma + \beta)\sin(\gamma - \beta)\sin(\alpha + \delta)\sin(\alpha - \delta)\\
&\qquad + \cos(\gamma + \beta)\cos(\gamma - \beta)\cos(\alpha + \delta)\cos(\alpha - \delta).
\end{aligned}$$

CHAPTER V.

FORMULAE RELATIVE TO TRIANGLES.

SECTION I.—RELATION BETWEEN THE ELEMENTS OF A RIGHT-ANGLED TRIANGLE.

111. *In a right-angled triangle any side is equal to the rectangle contained by the hypotenuse and the sine of the opposite, or the cosine of the adjacent angle.*



DEM.—Let ABC be a triangle, having the angle C right; then (§ 14),

$$BC = AB \,.\, \sin A, \quad AC = AB \,.\, \cos A.$$

Hence, denoting the numerical lengths of the sides BC, CA, AB by the letters a, b, c, respectively, we have

$$a = c \sin A, \quad b = c \cos A. \qquad (303)$$

112. *In a right-angled triangle, either side divided by the other is equal to the tangent of the opposite angle.*

DEM.—Dividing the equations (303) by each other, we get

$$\frac{a}{b} = \tan A, \text{ or } a = b \tan A. \qquad (304)$$

Cor.—$a = c \cos B, \quad b = c \sin B, \quad a = b \cot B. \qquad (305)$

Observation.—The three equations (303)–(304) are sufficient for the solution of all the cases of right-angled triangles. It will be seen that they have remarkable analogues in Spherical Trigonometry.

EXERCISES.—XXI.

1. Prove $$\tan^2(45 - \tfrac{1}{2}A) = \left(\frac{c-a}{c+a}\right) = \tan^2 \tfrac{1}{2}B.$$

2. ,, $$\tan^2(45 + \tfrac{1}{2}A) = \left(\frac{c+a}{c-a}\right) = \cot^2 \tfrac{1}{2}B.$$

3. ,, $$\sin 2A = \frac{2ab}{c^2}.$$

4. ,, $$\cos 2A = \frac{b^2 - a^2}{c^2}.$$

5. ,, $$\cos 4A = \frac{a^4 - 6a^2b^2 + b^4}{c^4}.$$

6. ,, $$\tan \tfrac{1}{2}A = \frac{a}{b+c}.$$

7. ,, $$\sin^2 \tfrac{1}{2}A = \frac{c-b}{2c}.$$

8. ,, $$\cos^2 \tfrac{1}{2}A = \frac{c+b}{2c}.$$

9. Prove, if s denote the semiperimeter,

$$c = \frac{s}{\cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sqrt{2}}.$$

10. ,, $$c = \frac{s-a}{\cos \tfrac{1}{2}A . \sin \tfrac{1}{2}B \sqrt{2}} = \frac{s-b}{\sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \sqrt{2}} = \frac{s-c}{\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sqrt{2}}.$$

11. The perpendicular on the hypotenuse from the right angle divides it into segments which are inversely proportional to the squares of the sines of the adjacent angles.

12. In any triangle the altitude divides the base into segments which are proportional to the cotangents of the adjacent base angles.

13. In any triangle the altitude divides the vertical angle into segments whose sines are inversely proportional to the adjacent sides.

14. If the sides of a rectangle be a, b, express the angles of the rhombus formed by joining the middle points of its sides in terms of a and b.

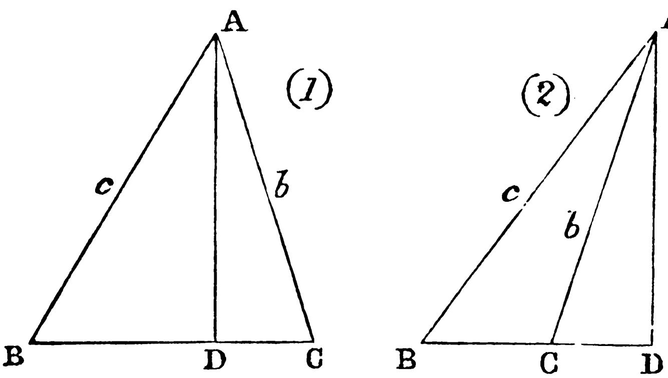
15. A tower and its spire subtend equal angles at a point whose distance from the foot of the tower is a; prove that if h be the altitude of the tower, the height of the spire is

$$\left(\frac{a^2 + h^2}{a^2 - h^2}\right) h.$$

Section II.—Oblique-Angled Triangles.

113. *In any plane triangle the sides are proportional to the sines of the opposite angles.*

This proposition has been already proved in § 36, *Cor.*



The following is the proof usually given:—Let ABC be any triangle. From A draw AD perpendicular to BC.

1°. Let B, C be acute angles. Then, from fig. (1), we have

$$AD = AB \sin B = AC \sin C, \text{ § } 111;$$

$$\therefore \quad b \sin C = c \sin B.$$

Hence $$b : c :: \sin B : \sin C.$$

2°. Let the angle C be obtuse. We have, from fig. (2),

$$AD = AC \sin ACD = AC \sin ACB,$$

since supplemental angles have equal sines, and

$$AD = AB \sin B.$$

Hence $$b \sin C = c \sin B, \text{ or } b : c :: \sin B : \sin C.$$

3°. If the angle C be right we have, from (305),

$$b : c :: \sin B : 1; \text{ but } 1 = \sin 90^\circ = \sin C;$$

$$\therefore \quad b : c :: \sin B : \sin C.$$

Hence, in every case, the sides are proportional to the sines of the opposite angles;

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{306}$$

114. *The sum of any two sides of a plane triangle is to the third side as the cosine of half the difference of the opposite angles is to the sine of half the remaining angle.*

Dem.—We have

$$\frac{a}{c} = \frac{\sin A}{\sin C}, \quad \frac{b}{c} = \frac{\sin B}{\sin C} \quad (\S\ 113);$$

$$\therefore \quad \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \sin \frac{1}{2} C \cos \frac{1}{2} C}.$$

But since $\frac{1}{2}(A+B)$ is the complement of $\frac{1}{2} C$,

$$\sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2} C.$$

Hence
$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}. \tag{307}$$

115. *The difference of any two sides of a triangle is to the third side as the sine of half the difference of the opposite angles is to the cosine of half the remaining angle.*

Dem.—From (306) we get

$$\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2} C \cos \frac{1}{2} C}.$$

But
$$\cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} C;$$

$$\therefore \quad \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}. \tag{308}$$

116. *The sum of any two sides of a triangle is to their difference as tan half the sum of the opposite angles is to tan half the difference.*

DEM.—From (306) we get

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \qquad (309)$$

We get the same result by dividing (307) by (308).

EXERCISES.—XXII.

1. Prove $$\frac{a+b+c}{c} = \frac{2 \cos \frac{1}{2} A \cos \frac{1}{2} B}{\sin \frac{1}{2} C}. \qquad (310)$$

2. „ $$\frac{a-b+c}{c} = \frac{2 \sin \frac{1}{2} A \cos \frac{1}{2} B}{\cos \frac{1}{2} C}. \qquad (311)$$

3. If the angles A, B, C of a triangle be equal to 2α, 4α, 8α, respectively, prove $(a+b+c) : c :: 1 : 2 \sin \alpha$. (Make use of Ex. 1).

4. If AD bisect the angle BAC, prove $BD : DC :: \sin C : \sin B$.

5. In the same case, prove $DC = b \sin \frac{1}{2} A \sec \frac{1}{2}(C - B)$.

6. If AD' bisect the external vertical angle, prove

$$BD' : CD' :: \sin C : \sin B.$$

7. In the same case, prove $CD' = b \cos \frac{1}{2} A \operatorname{cosec} \frac{1}{2}(C - B)$.

8. The vertical angle of a triangle is divided by the median that bisects the base into segments whose sines are inversely proportional to the adjacent sides.

9. If AD bisect BC, prove $\tan ADB = \pm \dfrac{2bc \sin A}{b^2 - c^2}$.

10. In the same case, prove

$$\cot DAB + \cot CAD = 4 \cot A + \cot B + \cot C.$$

117. *In every triangle, each side is equal to the sum of the products of the other sides into the cosines of the angles which they make with the first.*

DEM. (fig. (1), § 113)—We have

$$BC = BD + DC = AB \cos B + AC \cos C;$$

that is, $$a = c \cos B + b \cos C.$$

And from fig. (2) we have

$$BC = BD - CD = AB \cos B - AC \cos (\pi - C)$$
$$= AB \cos B + AC \cos C,$$

the same as before.

Thus, $$a = b \cos C + c \cos B, \quad (312)$$

$$b = c \cos A + a \cos C, \quad (313)$$

$$c = a \cos B + b \cos A. \quad (314)$$

EXERCISES.—XXIII.

1. Prove $(a \cos B - b \cos A) c = a^2 - b^2$.

2. ,, $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = (a + b + c)$.

3. ,, $a \cos A + b \cos B + c \cos C$
$= \frac{1}{2} \{a \cos (B - C) + b \cos (C - A) + c \cos (A - B)\}$.

4. ,, $a (\cos B \cos C + \cos A) = b (\cos C \cos A + \cos B$
$= c (\cos A \cos B + \cos C)$.

5. ,, $\cos A + \cos B = \dfrac{2(a+b)}{c} \sin^2 \frac{1}{2} C$.

6. ,, $\cos A - \cos B = \dfrac{2(b-a)}{c} \cos^2 \frac{1}{2} C$.

7. ,, $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$.

8. ,, $(a + b + c)(\cos A + \cos B + \cos C)$
$= 2 (a \cos^2 \frac{1}{2} A + b \cos^2 \frac{1}{2} B + c \cos^2 \frac{1}{2} C)$.

9. ,, $a \sin B \sin C = a \cos B \cos C + b \cos C \cos A + c \cos A \cos B$.

10. ,, $a^2 + b^2 + c^2 = 2 (ab \cos C + bc \cos A + ca \cos B)$.

11. ,, $a^2 + b^2 - c^2 : a^2 - b^2 + c^2 :: \tan B : \tan C$.

12. ,, $(a^2 - b^2) \cot C + (b^2 - c^2) \cot A + (c^2 - a^2) \cot B = 0$.

118. *In any plane triangle, the excess of the sum of the squares of any two sides over the square of the third side is equal to twice their product into the cosine of their included angle.*

Dem.—We have (fig. (1), § 113), from Euc. II., xiii.,

$$AB^2 = BC^2 + CA^2 - 2BC \,.\, CD;$$

but

$$CD = AC \cos C.$$

Hence

$$AB^2 = BC^2 + CA^2 - 2BC \,.\, AC \cos C;$$

or,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Again, suppose C to be obtuse (fig. (2), § 113), we have, from Euc. II., xii.,

$$AB^2 = BC^2 + CA^2 + 2BC \,.\, CD;$$

but

$$CD = AC \cos(\pi - C) = - AC \cos C.$$

Hence

$$AB^2 = BC^2 + CA^2 - 2BC \,.\, AC \cos C,$$

as before.

Thus

$$c^2 = a^2 + b^2 - 2ab \cos C, \tag{315}$$

$$a^2 = b^2 + c^2 - 2bc \cos A, \tag{316}$$

$$b^2 = c^2 + a^2 - 2ca \cos B. \tag{317}$$

119. From §§ 113, 117, 118, we have, between the six elements of a triangle, the three following groups of relation :—

$$\text{I.} \quad \begin{cases} \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}, \\ A + B + C = \pi. \end{cases}$$

$$\text{II.} \quad \begin{cases} a = b \cos C + c \cos B, \\ b = c \cos A + a \cos C, \\ c = a \cos B + b \cos A. \end{cases}$$

$$\text{III.} \quad \begin{cases} a^2 = b^2 + c^2 - 2bc \cos A, \\ b^2 = c^2 + a^2 - 2ca \cos B, \\ c^2 = a^2 + b^2 - 2ab \cos C. \end{cases}$$

Now, a triangle is determined by any three elements (except the three angles). Hence there can exist only three distinct equations between the elements; and therefore each of the groups I., II., III., must follow as a consequence from either of the other two. Thus, to infer I. from III. we have, from equation (316),

$$4b^2c^2\cos^2 A = (b^2 + c^2 - a^2)^2,$$

that is,

$$4b^2c^2 - 4b^2c^2\sin^2 A = 2b^2c^2 - 2c^2a^2 - 2a^2b^2 + b^4 + c^4 + a^4.$$

Hence $$\frac{\sin^2 A}{a^2} = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{4a^2b^2c^2}. \qquad (318)$$

And since the second side of this equation is unaltered by interchange of letters, we have

$$\frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2},$$

which is the same as group I.

Group II. is inferred from III. by adding two equations of III. Thus, by adding the second and third of III., and dividing by $2a$, we get the first of II.

From II. we can infer III. Thus, multiplying the equations II. by a, b, c, and subtracting each product from the sum of the other two.

In order to infer $A + B + C = \pi$ from II., we eliminate a, b, c. Thus we get

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C - 1 = 0;$$

or,

$$4\cos\frac{A+B+C}{2}\cos\frac{B+C-A}{2}\,.\cos\frac{C+A-B}{2}\cos\frac{A+B-C}{2} = 0.$$

(Equation 184.)

One factor of which, $\cos\dfrac{A+B+C}{2} = 0,$

gives $$A + B + C = \pi.$$

In a similar manner, from I. we can infer II. Thus—

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hence

$$\frac{a}{\sin A} = \frac{b \cos C + C \cos B}{\sin B \cos C + \cos B \sin C} = \frac{b \cos C + c \cos B}{\sin (B + C)}.$$

Hence $$a = b \cos C + c \cos B.$$

Exercises XXIV.

I. If the angle A of a triangle be equal to $\frac{2\pi}{3}$, prove $a^2 = b^2 + bc + c^2$.

2. If the angle A of a triangle be equal to $\frac{\pi}{3}$, prove $a^2 = b^2 - bc + c^2$.

3. In any triangle, prove

$$a^2 - b^2 = 2c \{a \cos (60^\circ + B) - b \cos (60^\circ + A)\}.$$

4. In any triangle, prove perimeter

$$2a \operatorname{cosec} \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

5. If ϕ be an auxiliary angle such that $\tan^2 \phi = \dfrac{4ab \sin^2 \frac{1}{2}C}{(a - b)^2}$,

prove $$c = (a - b) \sec \phi. \tag{319}$$

6. If $$\sin^2 \psi = \frac{4ab \cos^2 \frac{1}{2}C}{(a + b)^2}, \quad \text{prove} \quad c = (a + b) \cos \psi. \tag{320}$$

7. If a, b, be two adjacent sides of a parallogram, θ their included angle, prove difference of squares of diagonals $= 4ab \cos \theta$.

8. In any triangle, $c^2 = (a + b)^2 \sin^2 \frac{1}{2}C + (a - b)^2 \cos^2 \frac{1}{2}C$.

9. If a', b', c' be the medians of the triangle ABC, and A', B', C' the angles of a triangle whose sides are equal to a', b', c', prove that

$$4 (a'b' \cos C' + b'c' \cos A' + c'a' \cos B') = 3 (ab \cos C + bc \cos A + ca \cos B).$$

120. *To express the sine, the cosine, and the tangent of half an angle of a triangle in terms of the sides.*

1°. We have, from (316),

$$2bc\cos A = b^2 + c^2 - a^2, \qquad (\alpha)$$

and

$$2bc = 2bc. \qquad (\beta)$$

Hence $2bc(1-\cos A) = a^2 - (b-c)^2 = (a+b-c)(a-b+c)$.

Let $a+b+c=2s$, so that s is the semiperimeter of the triangle; then $(a-b+c) = 2(s-b)$, $(a+b-c) = 2(s-c)$.

Hence
$$4bc\sin^2\tfrac{1}{2}A = 4(s-b)(s-c); \qquad (322)$$

therefore
$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}. \qquad (323)$$

Similarly,
$$\sin\tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{ca}}, \qquad (324)$$

and
$$\sin\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

2°. By adding equations (α) and (β), we get

$$2bc(1+\cos A) = (b+c)^2 - a^2 = 2s \,.\, 2(s-a);$$

therefore
$$\cos\tfrac{1}{2}A = \sqrt{\frac{s\,.\,(s-a)}{bc}}. \qquad (325)$$

Similarly,
$$\cos\tfrac{1}{2}B = \sqrt{\frac{s\,.\,(s-b)}{ca}}, \qquad (326)$$

and
$$\cos\tfrac{1}{2}C = \sqrt{\frac{s\,.\,(s-c)}{ab}}. \qquad (327)$$

3°. By dividing (322) by (325),

we get $$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s\,.\,s-a}}. \tag{328}$$

Similarly, $$\tan \tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s\,.\,s-b}}, \tag{329}$$

and $$\tan \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s\,.\,(s-c)}}. \tag{330}$$

4°. By taking twice the product of (322) and (325), we get

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \tag{331}$$

Exercises.—XXV.

1. Prove $\tan \frac{1}{2}A \,.\, \tan \frac{1}{2}B = \dfrac{s-c}{s}$. (332)

2. ,, $\cos^2 \frac{1}{2}A : \cos^2 \frac{1}{2}B :: a(s-a) : b(s-b)$. (333)

3. ,, $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = 16s\,.\,(s-a)(s-b)(s-c)$. (334)

4. ,, $s - a = b\sin^2 \frac{1}{2}C + c\sin^2 \frac{1}{2}B$.

5. ,, $\dfrac{a^2-b^2}{c^2} = \dfrac{\sin(A-B)}{\sin C}$.

6. ,, $\dfrac{a^2-b^2}{ac} = \dfrac{\sin(A-B)}{\sin A}$.

7. ,, $\cos \frac{1}{2}B \cos \frac{1}{2}C = \dfrac{s}{a} \,.\, \sin \frac{1}{2}A$.

8. ,, $\sin \frac{1}{2}B \,.\, \sin \frac{1}{2}C = \dfrac{s-c}{a} \sin \frac{1}{2}A$.

9. ,, $\tan \frac{1}{2}A / \tan \frac{1}{2}B = (s-b)/(s-c)$.

10. ,, $(\tan \frac{1}{2}A - \tan \frac{1}{2}B)/(\tan \frac{1}{2}A + \tan \frac{1}{2}B) = (a-b)/c$.

11. ,, $\cot C = \dfrac{(a+c)\tan \frac{1}{2}B + (a-c)\cot \frac{1}{2}B}{2c}$.

12. Prove $(b-c)\cot\frac{1}{2}A+(c-a)\cot\frac{1}{2}B+(a-b)\cot\frac{1}{2}C=0.$

13. ,, $s^2.\tan\frac{1}{2}A.\tan\frac{1}{2}B.\tan\frac{1}{2}C=\sqrt{s(s-a).(s-b)(s-c)}.$

14. ,, $a^2\sin 2B+b^2\sin 2A=2ab\sin C.$

15. ,, $b\cos^2\frac{1}{2}C+c\cos^2\frac{1}{2}B=s.$

16. Prove that the lengths of the lines bisecting the angles of a triangle, and terminated by the opposite sides, are—

$$\frac{b\sin C}{\cos\frac{1}{2}(B-C)},\quad \frac{c\sin A}{\cos\frac{1}{2}(C-A)},\quad \frac{a\sin B}{\cos\frac{1}{2}(A-B)}.$$

17. If a^2, b^2, c^2 be in AP, prove that $\cot A$, $\cot B$, $\cot C$ are in AP.

18. If $\cot\frac{1}{2}B\cot\frac{1}{2}C=3$, prove that $b+c=2a$.

19. If a, b, c be in AP, and the greatest angle exceed the least by $90°$, prove that $a:b:c::\sqrt{7}-1:\sqrt{7}:\sqrt{7}+1.$

20. If $(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B)$, prove that the triangle is either isosceles or right-angled.

21. If a^2, b^2, c^2 be in AP, prove that

$$\sin 3B=\frac{a^2-c^2}{2ac}\sin B.$$

22. If the angles of a triangle form an AP, whose common difference is δ, prove that $$\cos\delta=\frac{a+c}{2b}.$$

23. If a, b, c be in AP, prove that

$$\cot\tfrac{1}{2}A,\ \cot\tfrac{1}{2}B,\ \cot\tfrac{1}{2}C \text{ are in } AP.$$

24. If a point O in the plane of a triangle be joined to the angular points A, B, C, prove that

$$\sin ABO.\sin BCO.\sin CAO=\sin OAB.\sin OBC.\sin OCA.$$

25. If C be a right angle, prove

$$\cos(2A-B)=\frac{a(3b^2-a^2)}{c^3}.$$

26. If x, y, z be three angles determined by the relations

$$\cos x = \frac{a}{b+c}, \quad \cos y = \frac{b}{c+a}, \quad \cos z = \frac{c}{a+b};$$

prove that—

1°. $\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2} = 1.$

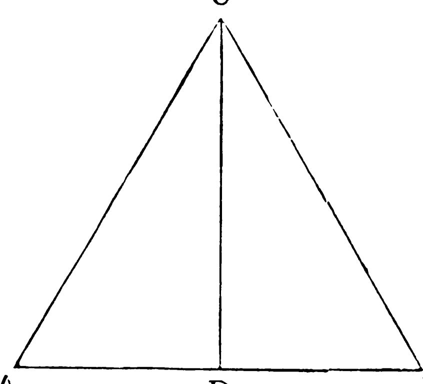
2°. $\tan\frac{x}{2} . \tan\frac{y}{2} \tan\frac{z}{2} = \tan\frac{A}{2} . \tan\frac{B}{2} \tan\frac{C}{2}.$

27. Prove that $bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2} + ab\cos^2\frac{1}{2}C = s^2.$

SECTION III.—AREA OF TRIANGLE.

121. To find Expressions for the Area of a Triangle.

1°. *The area of a triangle is equal to half the product of any two sides into the sine of their included angle.*



DEM.—Let ABC be the triangle, CD the perpendicular from C on AB. Then (EUC. II. I., *Cor.* 2), the area of

$$ABC = \tfrac{1}{2}AB . CD;$$

but $$CD = AC . \sin A.$$

Hence area $$= \tfrac{1}{2} AB . AC \sin A;$$

or, denoting the area by S,

$$S = \tfrac{1}{2}bc \sin A. \tag{335}$$

Cor. 1.—Area of a parallelogram whose adjacents are a, b

$$= ab \sin (\widehat{ab}). \qquad (336)$$

Cor. 2.—Area of a quadrilateral whose diagonals are $\delta\delta'$

$$= \tfrac{1}{2}\delta\delta' \sin (\widehat{\delta\,\delta'}). \qquad (337)$$

[Circumscribe a parallelogram whose sides are parallel to the diagonals.]

Cor. 3.—If R be the circumradius, area $= \dfrac{abc}{4R}$; (338)

because (§ 36), $\qquad \sin A = a/2R.$

2°. *The area in terms of the sides.*

We have in equation (331)

$$\sin A = \frac{2}{bc}\sqrt{s\,.\,(s-a)\,.\,(s-b)\,.\,(s-c)}.$$

Substituting this in (335), we get

$$S = \sqrt{s\,.\,(s-a)(s-b)(s-c)}. \qquad (339)$$

3°. If we substitute the value of b, got from the proportion

$$\sin B : \sin C :: b : c \text{ in } (335),$$

we get $\quad S = \tfrac{1}{2}c^2\,.\,\dfrac{\sin A\,.\,\sin B}{\sin C} = \dfrac{c^2}{2(\cot A + \cot B)}.$ (340)

4°. Substitute for b and c from

$$b/\sin B = c/\sin C = 2R$$

in (335), and we get

$$S = 2R^2 \sin A\,.\,\sin B\,.\,\sin C. \qquad (341)$$

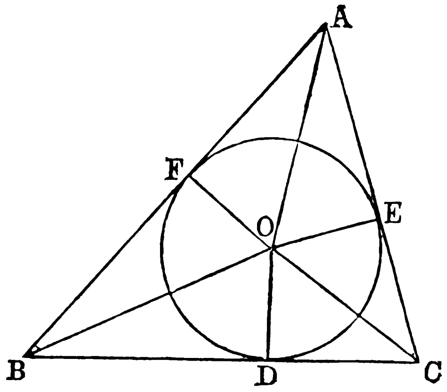
5°. $$S = s^2 . \tan\frac{A}{2} . \tan\frac{B}{2} . \tan\frac{C}{2}, \tag{342}$$

inferred from $$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ \&c.}$$

6°. $$S = \tfrac{1}{4}\sqrt{(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)}, \tag{343}$$

from (318) and (335).

7°. If r denote in-radius, $S = sr$.



For, if O be the in-centre, D, E, F the points of contact, the area of the triangle $AOB = \frac{cr}{2}$,

,, ,, $BOC = \frac{ar}{2}$,

,, ,, $COA = \frac{br}{2}$;

$$\therefore \quad S = \frac{a+b+c}{2} r = sr. \tag{344}$$

8°. $$S = r^2 . \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}, \tag{345}$$

inferred from 5° and 7°.

EXERCISES.—XXVI.

1. Prove $S = (s-a)^2 \tan \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$. (346)

2. ,, $S = \frac{1}{4}\left(\frac{a^2+b^2+c^2}{\cot A + \cot B + \cot C}\right)$. (347)

3. ,, $S = \frac{1}{4}(a^2 \cot A + b^2 \cot B + c^2 \cot C)$. (348)

4. ,, $S = R(a \cos C \cos A + b \cos A \cos B + c \cos B \cos C)$. (349)

5. ,, $S = \frac{ab}{8R^2}\left\{a\sqrt{4R^2-b^2} + b\sqrt{4R^2-a^2}\right\}$. (350)

6. If h', h'', h''' denote the altitudes, prove

$$S = \frac{R}{2s}(h'h'' + h''h''' + h'''h') \tag{351}$$

$$= \frac{R}{2(s-a)}(h'h'' - h''h''' + h'''h').$$

7. ,, $S = \sqrt{\tfrac{1}{2}(Rr)(h'h'' + h''h''' + h'''h')}$. (352)

8. Prove that the area of the orthocentric triangle

$$= 2S \cos A \cos B \cos C. \tag{353}$$

9. Prove area of triangle formed by points of contact of incircle

$$= S \sin\frac{A}{2} \cdot \sin\frac{B}{2} \sin\frac{C}{2} = 2r^2 \cos\frac{A}{2} \cdot \cos\frac{B}{2} \cos\frac{C}{2}. \tag{354}$$

10. If A', B', C' be the reflections of the summits A, B, C of the triangle ABC with respect to the opposite sides, prove that the area of

$$\Delta A'B'C' = S(3 + 8 \cos A \cos B \cos C). \tag{355}$$

11. If β', β'', β''' denote bisectors of angles, prove that

$$4S = \sqrt{\frac{\beta'\beta''\beta'''(a+b)(b+c)(c+a)}{2Rs}}. \tag{356}$$

12. ,, $S = \frac{Rr}{\beta'\beta''\beta'''}\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right)$. (357)

13. ,, $S = \frac{1}{2}a\beta' \sin(C + \frac{1}{2}A)$. (358)

14. ,, $S = 2s(\beta' \sin\frac{1}{2}A + \beta'' \sin\frac{1}{2}B + \beta''' \sin\frac{1}{2}C)$. (359)

15. ,, $S = c(\beta' \sin\frac{1}{2}A + \beta'' \sin\frac{1}{2}B - \beta''' \sin\frac{1}{2}C)$. (360)

16. ,, $4S = \beta'(h''+h''') \sec\frac{1}{2}A = \frac{h'(h''+h''')}{\cos\frac{1}{2}A \sin(C+\frac{1}{2}A)}$. (361)

17. If through the summits A, B, C of a triangle we draw three lines, making with AB, BC, CA the same angle α, prove that the area of the triangle formed by these lines is

$$S\sqrt{\frac{\sin(\omega+\alpha)}{\sin\omega}}, \tag{362}$$

where ω is the angle $\cot^{-1}(\cot A + \cot B + \cot C)$. (NEUBERG.)

18. If through the summits of a triangle we draw three lines making the same angle α with the opposite sides, prove that the area of the triangle formed by these lines is $4S\cos^2\alpha$. (*Ibid.*) (363)

19. If the adjacent sides of a parallelogram be a, b, and their included angle A, prove that the area of the parallelogram formed by the bisectors of ts interior angles is $\frac{1}{2}(a-b)^2\sin A$. (364)

20. If ABC be any triangle, M any point in its plane, the area of the triangle formed by the orthocentres of the triangles AMB, BMC, CMA is equal to the area of the triangle ABC. (NEUBERG.)

21. If e', e'', e''' be the reciprocals of the altitudes of ABC, prove

$$S = \frac{s}{e'+e''+e'''} = \frac{s-a}{e''+e'''-e'} = \&c. \tag{365}$$

22. If $2\Sigma = e'+e''+e'''$, prove

$$16S = \frac{1}{\sqrt{\Sigma\,.\,(\Sigma-e')\,.\,(\Sigma-e'')(\Sigma-e''')}}. \tag{366}$$

23. The area of the triangle whose sides are equal to the medians $= \frac{3}{4}S$. (367)

24. If m', m'', m''' be the medians of a triangle, prove

$$S = \sqrt{\frac{(m'^2+m''^2+m'''^2)/(e'^2+e''^2+e'''^2)}{3}}. \tag{368}$$

25. If μ, μ' denote the angles made with the base by the median and the bisector of the vertical angle, prove

$$4S = a^2\sin\mu\sqrt{\frac{\sin 2\mu'}{\sin 2(\mu'-\mu)}}. \tag{369}$$

SECTION IV.—FORMULAE RELATIVE TO RADII OF CIRCLES.

122. *To find the in-radius of a triangle in terms of the sides.*

We have (see fig., § 121, 7°), $rs = S$. (Equation 344.)

Hence $$rs = \sqrt{s(s-a)(s-b)(s-c)}. \quad \text{(Equation 339.)}$$

$$\therefore \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{369}$$

Cor. 1.—

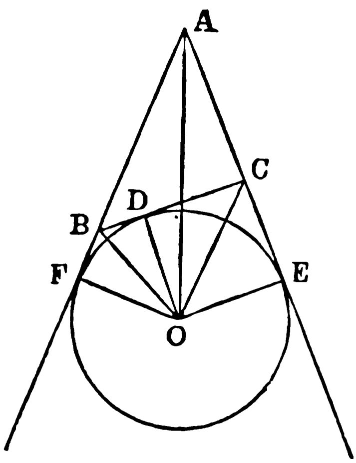
$$AF = (s-a) \text{ [“Sequel,” IV. I.]} = r \cot \frac{A}{2}. \tag{370}$$

$$BF = (s-b) \quad ,, \quad ,, \quad = r \cot \frac{B}{2}. \tag{371}$$

$$CE = (s-c) \quad ,, \quad ,, \quad = r \cot \frac{C}{2}. \tag{372}$$

Cor. 2.—$(s-a) : (s-b) : (s-c) : s :: \cot \frac{1}{2}A : \cot \frac{1}{2}B$
$: \cot \frac{1}{2}C : \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$. (373)

123. *To find the radii of the escribed circles.*



Denoting the radii of the escribed circles by r', r'', r''', we have area of triangle $BOC = \frac{1}{2}ar'$,

,, ,, $COA = \frac{1}{2}br'$,

,, ,, $AOB = \frac{1}{2}cr'$.

Hence, adding the two last, and subtracting the first, we get

$$S = \tfrac{1}{2}(b + c - a)r' = (s - a)r';$$

$$\therefore \quad r' = \frac{S}{s-a}. \tag{374}$$

In like manner, $$r'' = \frac{S}{s-b}, \tag{375}$$

and $$r''' = \frac{S}{s-c}. \tag{376}$$

Cor. 1.— $$\frac{1}{r} = \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}. \tag{377}$$

Cor. 2.— $$S = \sqrt{rr'r''r'''}. \tag{378}$$

Cor. 3.— $$s = \sqrt{r'r'' + r''r''' + r'''r'}. \tag{379}$$

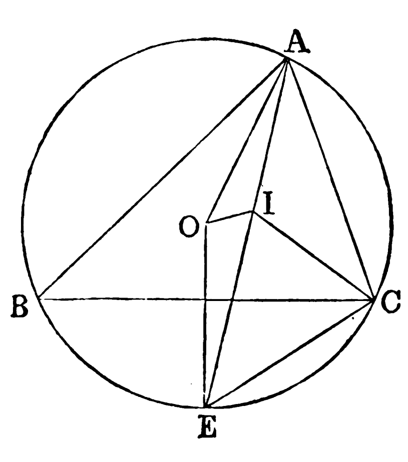
124. *To find the circumradius in terms of the sides.*

Let R = circumradius; then $S = \dfrac{abc}{4R}$. (Equation 338.)

Hence $$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \tag{380}$$

125. *To find the distance between the incentre and circumcentre of a triangle.*

Let I, O be the incentre and circumcentre, respectively.



Join AI, and produce to meet the circle in E. Join OA, OI, OE, CE. Now the angles EIC, ECI are evidently each equal to $\frac{1}{2}(A + C)$. Hence $EI = EC$; but (§ 36), $EC = 2R \sin \frac{1}{2}A$; $\therefore$ $EI = 2R \sin \frac{1}{2}A$. And since I is the incentre, $IA \,.\, \sin \frac{1}{2}A = r$. Hence $EI \,.\, IA = 2Rr$.

Now, denoting OI by δ, since the triangle AOE is isosceles, we have $OA^2 - OI^2 = EI \,.\, IA$ ("Sequel," II. 1).

Hence $\quad R^2 - \delta^2 = 2Rr$; $\quad$ or $\quad \dfrac{1}{r} = \dfrac{1}{R + \delta} + \dfrac{1}{R - \delta}$. $\qquad$ (381)

Exercises XXVII.

1. The sides of a triangle are 13, 14, 15. Find the radii of its circum-circle, and of its inscribed and escribed circles.

2. Prove $\quad 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = r$. $\qquad$ (382)

3. ,, $\quad r = a \sin \frac{1}{2}B \sin \frac{1}{2}C \sec \frac{1}{2}A$. $\qquad$ (383)

4. ,, $\quad r' = a \cos \frac{1}{2}B \cos \frac{1}{2}C \sec \frac{1}{2}A$. $\qquad$ (384)

5. ,, $\quad r'r''r''' = r^3 \cot^2 \dfrac{A}{2} \cot^2 \dfrac{B}{2} \cot^2 \dfrac{C}{2}$. $\qquad$ (385)

6. ,, $\quad 2R + 2r = a \cot A + b \cot B + c \cot C$. $\qquad$ (386)

7. If a_1, b_1, c_1 be the sides of the triangle formed by the centres of the escribed circles, prove $\quad a/a_1 = \sin \frac{1}{2}A$, &c. $\qquad$ (387)

8. The area of the same triangle $= abc/r$ $\qquad$ (388)

9. The distances of the incentre of a triangle from the centres of its escribed circles are

$$a \sec \frac{A}{2}, \quad b \sec \frac{B}{2}, \quad c \sec \frac{C}{2}.$$

10. If d, e, f be the distances of the incentre from A, B, C, prove

$$ad^2 + be^2 + cf^2 = abc,$$

and

$$d^2\left(\frac{1}{b} - \frac{1}{c}\right) + e^2\left(\frac{1}{c} - \frac{1}{a}\right) + f^2\left(\frac{1}{a} - \frac{1}{b}\right) = 0.$$

11. If p, p' be the perpendiculars from B, C on the bisector of the angle A, prove $\qquad pp' = (s - b)(s - c)$.

12. Prove that $R : r :: \sin A + \sin B + \sin C : 2 \sin A \sin B \sin C$.

13. If δ', δ'', δ''' denote the distances of the circumcentre from the centres of the escribed circles, prove

$$\frac{r'}{R+\delta'} + \frac{r'}{R-\delta'} = -1, \text{ \&c.} \tag{389}$$

14. Prove $$S = \frac{r}{2R^{\frac{1}{2}}} \sqrt{(r'+r'')(r''+r''')(r'''+r')}. \tag{390}$$

15. ,, $$S = \frac{r^2}{abc}(r'+r'')(r''+r''')(r'''+r'). \tag{391}$$

16. ,, $$S^2 = \left(a^2+b^2+c^2\right) \Big/ \left(\frac{1}{r^2}+\frac{1}{r'^2}+\frac{1}{r''^2}+\frac{1}{r'''^2}\right). \tag{392}$$

17. If O be the incentre of the triangle ABC, r_1, R_1 the inradius and circumradius of AOB, prove that

$$r_1 = c/(\cot\tfrac{1}{4}A + \cot\tfrac{1}{4}B), \quad R_1 = c/(2\cos\tfrac{1}{2}C).$$

18. Prove that the angles of the orthocentric triangle are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$, respectively.

19. Prove that the lengths of its sides are $a\cos A$, $b\cos B$, $c\cos C$.

20. Prove that its perimeter is $4R\sin A\sin B\sin C$.

21. Prove $S = arr'/(r'-r) = ar''r'''/(r''+r''')$.

22. ,, $S = rr'(r''+r''')/a = r''r'''(r'-r)/a$.

23. ,, $S = (a+b)rr'''/(r+r''') = (a-b)r'r''/(r'-r'')$.

24. ,, $S = rr'(r''-r''')/(b-c) = r''r'''(r+r')/(b+c)$.

25. ,, $S = rr'\sqrt{(r''+r''')}/\sqrt{(r'-r)} = r''r'''\sqrt{(r'-r)}/\sqrt{(r''+r''')}$.

26. ,, $S = sh'r'/(h'+2r') = (s-a)h'r/(h'-2r)$.

27. ,, $S = \sqrt{rr'/(e'+e''-e''')(e'-e''+e''')}$.

28. ,, $S = \sqrt{r''r'''/(e'+e''+e''')(e''+e'''-e')}$.

29. ,, $S = (r'+r''+r'''-r)^2 h'h''h'''/(8abc)$.

30. ,, $S = \frac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C)$.

CHAPTER VI.

RESOLUTION OF TRIANGLES AND QUADRILATERALS.

126. Every triangle has six parts, namely, three sides and three angles. When any three of these are given, except the three angles, the remaining parts can be calculated, the process of doing which is called the *solution of triangles*. The reason that the three angles are insufficient is, that they are not independent; for (Euc. I., xxii.), if two of them be given, the third is determined. Triangles are divided in Trigonometry into right-angled and oblique. We shall commence with the solution of the former.

Section I.—The Right-angled Triangle.

127. There are four cases of right-angled triangles:—

I. Given one side and the hypotenuse.

II. ,, two sides.

III. ,, one side and one of the acute angles.

IV. ,, the hypotenuse and one of the acute angles.

We shall solve these cases—1°. Without logarithms; 2°. By logarithms.

The following are the formulae for the solution of right-angled triangles:—

$$\sin A = \frac{a}{c}, \quad \sin B = \frac{b}{c}, \tag{393}$$

$$\cos A = \frac{b}{c}, \quad \cos B = \frac{a}{c}, \tag{394}$$

$$\tan A = \frac{a}{b}, \quad \tan B = \frac{b}{a}. \tag{395}$$

The second set are got from the first by interchange of letters.

EXAMPLE 1.—

Given $a = 119{\cdot}7070, \quad c = 3500$; find b, α, β.

We have, from (393),

$$\sin A = \frac{a}{c} = \frac{119{\cdot}7070}{3500} = {\cdot}3420200;$$

$$\therefore \quad A = 20^\circ. \quad \text{Hence} \quad B = 70^\circ.$$

And from (394),

$$b = c \cos A = 3500 \times {\cdot}9396926 = 3288{\cdot}9241.$$

EXAMPLE 2.—

Given $a = 225{\cdot}101, \quad b = 250$; find C, A, B.

From (395), $\quad \tan A = \dfrac{a}{b} = \dfrac{225{\cdot}101}{250} = {\cdot}9004040;$

$$\therefore \quad A = 42^\circ, \quad \text{and} \quad B = 48^\circ.$$

Again, from (394), we get

$$c = b \sec A = 250 + 1{\cdot}3456327 = 336{\cdot}4082.$$

EXAMPLE 3.—

Given $a = 240, \quad A = 23^\circ\ 8'$; find B, c, b.

Since $A + B = 90^\circ$, we have $B = 66^\circ\ 52'$.

And from (393), we get

$$c = a \operatorname{cosec} A = 240 \times 2{\cdot}5453561 = 610{\cdot}87546.$$

Also, from (395),

$$b = a \cot A = 240 \times 2{\cdot}3469028 = 563{\cdot}2567.$$

EXAMPLE 4.—

Given $c = 560, \quad A = 35^\circ\ 32'$; find B, a, b.

From (393),

$$a = c \sin A = 560 \times {\cdot}5811765 = 324{\cdot}5084.$$

From (394),

$$b = c \cos A = 560 \times {\cdot}8137775 = 455{\cdot}7154.$$

Exercises.—XXVIII.

1. A May-pole was broken by the wind, and its top struck the ground 20 feet from the base; had it been broken 5 feet lower down its top would have extended 10 feet further from its base. Required the height.

2. A tower 75 feet high casts a shadow of 45 feet on the horizontal plane on which it stands. Find the sun's altitude.

3. An object $7\frac{1}{2}$ feet high, placed on the top of a tower, subtends an angle whose tangent is ·015, at a place whose horizontal distance from the foot of the tower is 125 feet. Find the height of the tower.

4. The peak of Teneriffe is $2\frac{1}{2}$ miles high, and the angular depression of the horizon from its summit is $2^\circ\ 1'\ 47''$. Find the earth's diameter.

128. Calculation by Logarithms.

Case I.—*Given a, c, it is required to find b, A, B.*

From the equation $\sin A = \frac{a}{c}$ (393), by taking logarithms, we get

$$L \sin A = 10 + \log a - \log c. \qquad (396)$$

This determines A; then $B = 90^\circ - A$,

and (Euc. I., xlvii.), $b^2 = (c + a)(c - a)$.

Hence
$$\log b = \tfrac{1}{2}\{\log(c + a) + \log(c - a)\}. \qquad (397)$$

Or b may be found from (394), which gives

$$\log b = \log c + L \cos A - 10. \qquad (398)$$

Example.—Given $a = 21$, $c = 29$; find b, A, B.

Type of the Calculation.

$$c + a = 50, \qquad c - a = 8.$$

$\log b = \frac{1}{2}\{\log(c+a) + \log(c-a)\}$,	$\log a = 1{\cdot}3222913$,
$\log(c + a) = 1{\cdot}6989700$,	$\log c = 1{\cdot}4623980$,
$\log(c - a) = {\cdot}9030900$.	$L \sin A = 9{\cdot}8598213$.
Hence	Hence
$\log b = 1{\cdot}3010300$; $\therefore b = 20$.	$A = 46^\circ 23' 50''$, $B = 43^\circ 36' 10''$.

If the difference $c - a$ be very small when compared to c, the angle A is very near 90°, and cannot be determined with much precision from the formula $\sin A = \frac{a}{c}$. In this case it is preferable to employ either of the following:—

$$\sin \tfrac{1}{2}B = \sqrt{\frac{1 - \cos B}{2}} = \sqrt{\frac{c - a}{2c}}, \tag{399}$$

$$\tan \tfrac{1}{2}B = \sqrt{\frac{1 - \cos B}{1 + \cos B}} = \sqrt{\frac{c - a}{c + a}}. \tag{400}$$

Observation.—For the logarithms of the trigonometrical functions French authors use the logarithms of the Tables diminished by 10. Thus

$$\log \sin A = \bar{1}{\cdot}8598213,$$

instead of

$$\text{L} \sin A = 9{\cdot}8598213.$$

There is no doubt but this is the better way, and will be soon adopted by English mathematicians; but we do not wish to be the first to make the innovation.

Case II.—*Given a, b, it is required to find A, B, c.*

From (395), we get

$$\text{L} \tan A = 10 + \log a - \log b. \tag{401}$$

Hence A is determined, and then

$$B = 90^\circ - A.$$

Again, from (393), we have

$$\log c = 10 + \log a - \text{L} \sin A. \tag{402}$$

Or c may be found from

$$c^2 = a^2 + b^2.$$

Example.—

Given $\quad a = 2266{\cdot}35, \quad b = 5439{\cdot}24$; find A, B, c.

Type of the Calculation.

$\log a = 3{\cdot}3553270,$	$\mathrm{L}\sin A = 9{\cdot}5850266;$
$\log b = 3{\cdot}7355382.$	but $\log c = 10 + \log a - \mathrm{L}\sin A;$
Hence $\mathrm{L}\tan A = 9{\cdot}6197888;$	$\therefore \quad \log c = 3{\cdot}7703004.$
$\therefore \quad A = 22^\circ\ 37'\ 11''{\cdot}5,$	Hence $c = 5892{\cdot}51.$
and $B = 67^\circ\ 22'\ 48''{\cdot}5.$	

Case III.—*Given a, A, it is required to find B, b, c.*

From (402), we have

$$\log c = 10 + \log a - L \sin A.$$

From (401), we have

$$\log b = 10 + \log a - L \tan A. \qquad (403)$$

Lastly, $B = 90^\circ - A$. Hence c, b, B are determined.

Example.—

Given $a = 5472{\cdot}5$, $A = 32^\circ\ 15'\ 24''$; find B, b, c.

Type of the Calculation.

$10 + \log a = 13{\cdot}7381858,$	$10 + \log a = 13{\cdot}7381858,$
$\mathrm{L}\sin A = 9{\cdot}7273076.$	$\mathrm{L}\tan A = 9{\cdot}8001090.$
Hence $\log c = 4{\cdot}0108782;$	Hence $\log b = 3{\cdot}9380768;$
$\therefore \quad c = 10253{\cdot}9.$	$\therefore \quad b = 8671{\cdot}5.$

Case IV.—*Given c, A, it is required to find a, b, B.*

From (398), we get

$$\log a = \log c + L \sin A - 10, \qquad (404)$$

and B, b may be calculated as in Case II.

Exercises.—XXIX.

1.	Given	$a = 324$,	$c = 640$;	find	b, A, B.
2.	,,	$a = 3$ furlongs 12 perches,	$c = 1\cdot5$ miles;	,,	b, A, B.
3.	,,	$a = 95\cdot42$ feet,	$b = 40\cdot45$ yards;	,,	c, A, B.
4.	,,	$a = 3\cdot65$ metres,	$b = 4\cdot25$ metres;	,,	c, A, B.
5.	,,	$a = 4\cdot42$ metres,	$A = 35°$;	,,	B, c, b.
6.	,,	$a = 17\cdot34$ yards,	$B = 69° 30'$;	,,	A, c, b.
7.	,,	$c = 1$ mile,	$A = 54° 8'$;	,,	B, a, b.

Section II.—Oblique-angled Triangles.

129. There are four cases of oblique-angled triangles.—

I. A side and two angles.

II. Two sides, and the angle opposite to one of them.

III. Two sides and the included angle.

IV. The three sides.

Case I.—*Suppose B, C are the given angles, and a the given side.*

Then $A = 180° - (B + C)$. Hence A is determined.

Again,
$$\frac{b}{a} = \frac{\sin B}{\sin A}, \quad \frac{c}{a} = \frac{\sin C}{\sin A}.$$

Hence
$$\log b = \log a + \text{L} \sin B - \text{L} \sin A, \qquad (405)$$

,,
$$\log c = \log a + \text{L} \sin C - \text{L} \sin A. \qquad (406)$$

Hence the required parts are found.

If A, B be the given angles, we have

$$C = 180° - (A + B),$$

and b, c can be found as before.

EXAMPLE.—

Given $B = 38° \ 12' \ 48''$, $C = 60°$, $a = 7012{\cdot}6$; find b, c.

Type of the Calculation.

$A = 180° - (B + C)$;

$\therefore A = 81° \ 47' \ 12''$,

$\log a = 3{\cdot}8458729$,

$\text{L} \sin B = 9{\cdot}9714038$,

$\text{L} \sin A = 9{\cdot}9955225$;

$\therefore \log b = 3{\cdot}6417542.$

Hence $b = 557815.$

$\text{L} \sin C = 9{\cdot}9375306.$

Hence $\log c = 3{\cdot}7878810$;

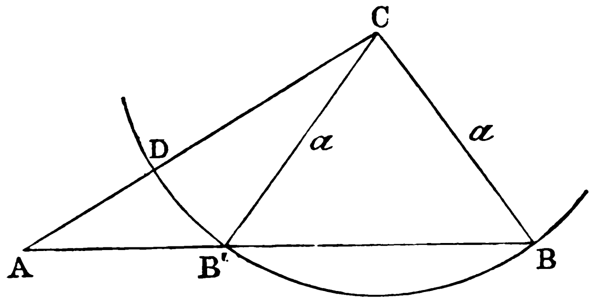
$\therefore c = 6135{\cdot}94.$

130. CASE II.—*Given* a, b, *and the angle* A, *to solve the triangle.*

From (405), we get

$$\text{L} \sin B = \text{L} \sin A + \log b - \log a. \tag{407}$$

Since the sine of an angle is equal to the sine of its supplement, when an angle is found from its sine, it is sometimes doubtful which of the supplemental angles having the given sine is to be selected. *This happens only when two sides and the angle opposite to the less are given.* Thus, if ACB be a triangle, having AC greater than CB, and if CD be cut off equal to CB, the circle



described with C as centre, and CD as radius, will cut the base AB in two points B, B'; then joining CB', we have two triangles, viz., CAB, CAB', having the parts b, a, A, exactly the same in both, although their remaining parts are different.

Hence each of these triangles fulfils the required conditions, on which account this case of the solution of triangles is called the *ambiguous case.*

When two sides and the angle opposite to the greater are given, there can be no ambiguity, for the angle opposite to the less must be acute.

131. The angle B having been determined, C is given by the equation

$$A + B + C = 180°,$$

and the third side can be calculated as in Case I. It can also be found as follows:—

We have $$b^2 + c^2 - 2b \cos A = a^2;$$

and, solving as a quadratic, we get

$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}. \qquad (408)$$

The part affected by the radical will evidently vanish if the angle B be right. In this case there will be no ambiguity, and we have $c = b \cos A$. If a be less than $b \sin A$, the values of c will be imaginary, and there will be no triangle answering to the given conditions. Lastly, if a be $> b \sin A$, there will be two real values for c. In order that each may be positive, $b^2 \cos^2 A$ must be $> a^2 - b^2 \sin^2 A$, or $a < b$. *Hence there will be no ambiguity, except when* $a < b$, *and* $> b \sin A$.

132. In order to render the formula (408), calculable by logarithms, we write it

$$c = b \cos A \pm b \sin A \sqrt{\left(\frac{a^2}{b^2 \sin^2 A} - 1\right)};$$

or, putting $$a/b \sin A = \operatorname{cosec} \phi,$$

$$c = (b \cos A \pm b \sin A \cot \phi) = b \sin (\phi \pm A) \operatorname{cosec} \phi.$$

Hence $$c = a \sin (\phi \pm A) \operatorname{cosec} A. \qquad (409)$$

EXAMPLE.—
Given $a = 7$, $b = 8$, $A = 27° 47' 45''$; find B, C, c.

Type of the Calculation.

$$\text{L} \sin B = \text{L} \sin A + \log b - \log a,$$

$$\text{L} \sin A = 9\cdot6686860,$$

$$\log b = \cdot9030900,$$

$$\log a = \cdot8450980;$$

$$\therefore \quad \text{L} \sin B = 9\cdot7266780.$$

There are two solutions:—

$B = 32° 12' 15''$,

or $B = 147° 47' 45''$.

Hence

$C = 120°$, or $C = 4° 24' 30''$,

and $c = 13$, or $c = 1\cdot15385$.

133. CASE III.—*Given a, b, and the angle C, to find A, B, c.*

From § 116, we have

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$

Hence

$$\text{L} \tan \tfrac{1}{2}(A - B) = \text{L} \tan \tfrac{1}{2}(A + B) + \log(a - b) - \log(a + b). \quad (410)$$

Now, since C is given, its supplement $A + B$ is given, and equation (410) determines $A - B$. Hence A, B are found; the side c can be found (§ 114), which gives

$$\log c = \log(a + b) + \text{L} \sin \tfrac{1}{2} C - L \cos \tfrac{1}{2}(A - B). \quad (411)$$

The side c may also be found by means of either of the auxiliary ϕ, ψ (Exercises XXIV., 5, 6).

Should the case occur that a, b were known only by their logarithms, then we should write

$$\tan \tfrac{1}{2}(A - B)/\tan \tfrac{1}{2}(A + B) = \frac{a - b}{a + b} = \frac{1 - \dfrac{b}{a}}{1 + \dfrac{b}{a}} = \tan(45 - \phi)$$

if $$\frac{b}{a} = \tan \phi,$$

or $$\tan \tfrac{1}{2}(A - B) = \tan \tfrac{1}{2}(A + B) \tan(45 - \phi). \quad (412)$$

134. Case IV.—*Given the three sides of a triangle, to find the angles.*

We have (§ 122), $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$,

and (§ 122, *Cor.* 1), $\tan \frac{1}{2}A = \dfrac{r}{s-a}$;

$$\therefore \quad \log r = \tfrac{1}{2}\{\log(s-a) + \log(s-b) + \log(s-c) - \log s\}, \quad (413)$$

$$\mathrm{L} \tan \tfrac{1}{2}A = 10 + \log r - \log(s-a). \quad (414)$$

Similarly, $$\mathrm{L} \tan \tfrac{1}{2}B = 10 + \log r - \log(s-b), \quad (415)$$

and $$\mathrm{L} \tan \tfrac{1}{2}C = 10 + \log r - \log(s-c). \quad (416)$$

Case IV. can also be solved by dividing the given triangle into two right-angled triangles by a perpendicular. Thus the perpendicular from A on BC divides it into two segments, BD, DC, which are respectively equal to

$$\frac{a^2 + b^2 - c^2}{2a}, \quad \text{and} \quad \frac{a^2 + c^2 - b^2}{2a}. \quad (417)$$

And then we have two right-angled triangles, in each of which a hypotenuse and a side are given.

Exercises.—XXX.

1. Given $a = 7$, $b = 8$, $C = 120$; find A, B, c.
2. ,, $a = 516$, $b = 219$, $C = 98° 54''$; ,, A, B, c.
3. ,, $a = 13$, $b = 14$, $c = 15$; ,, r and A, B, C.
4. ,, $A = 18°$, $a = 3$, $b = 3 + \sqrt{5}$; ,, B, C, c.
5. ,, $A = 15°$, $a = 5$, $b = 5(1 + \sqrt{3})$; ,, B, C, c.
6. Prove that the altitudes of a triangle are inversely proportional to

$$(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C), \quad (\cot \tfrac{1}{2}C + \cot \tfrac{1}{2}A), \quad (\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B).$$

7. If the angles of a triangle be in the ratio 1 : 2 : 7, prove that the greatest side : the least : : $\sqrt{5}+1 : \sqrt{5}-1$.

8. The angles of a triangle are as 1 : 2 : 3, and the difference between the greatest and the least side is 1000 yards; find the sides.

9. Given $a = 18$, $b = 2$, $C = 55°$; find A, B, being given $\log 2 = \cdot 3010300$, $L \tan 62° 30' = 10\cdot 2835233$, $L \tan 56° 56' = 10\cdot 1863769$, tab. diff. for 1' $= 2763$.

10. If $A = 30°$, $a = 3$, $b = 3\sqrt{3}$, prove $C = 90°$.

11. Prove that the line which divides one of the angles of an equilateral triangle in the ratio 3 : 1 divides the opposite side in the ratio

$$\sqrt{3}+1 : 2.$$

12. If C, C' be the values of the third angle in the ambiguous case when a, b, A are given, and $b > a$, prove

$$\tan A = \cot \tfrac{1}{2}(C + C').$$

13. Given $L \sin A = 9\cdot 9358921$, $L \sin 59° 37' 40'' = 9\cdot 9358894$, tab. diff. for $10'' = 124$, find A.

14. The sides a, b are 9, 7, $C = 64° 12'$; find A, B, being given $\log 2 = \cdot 3010300$, $L \tan 57° 54' = 10\cdot 2025255$, $L \tan 11° 16' = 9\cdot 2993216$, $L \tan 11° 17' = 9\cdot 2998804$.

15. If the angle A of a triangle be very obtuse, show that the sum of B and C is very nearly

$$206264\cdot 8 \sqrt{\frac{4s \,.\, s-a}{bc}} \text{ seconds.}$$

16. If the side of the base of a square pyramid be to the length of an edge : : 4 : 3, find slope of each face, being given $\log 2 = \cdot 3010300$, $L \tan 26° 33' = 9\cdot 6986800$, tab. diff. for $1' = 3200$.

17. The sides of a triangle are 18, 20, 22; find L tan of the least angle, being given

$$\log 2 = \cdot 3010300, \quad \log 3 = \cdot 4771213.$$

18. Given $a = 19$, $b = 1$, $C = 60°$; find A, B, being given

$$\log 3 = \cdot 4771213, \quad L \tan 57° 19' 11'' = 10\cdot 1928032.$$

19. The radii of two circles are r, r', and the distance between their centres is d; find—1°. The angle of intersection of their common tangents. 2°. The angle of intersection of the circles. 3°. The distance of a point of intersection of the circles from the intersection of the common tangents. 4°. The angle under which the intercept on one of the common tangents between the points of contact is seen from either point of intersection of the circles.

20. Find the radius of a circle inscribed in a lozenge whose side is a, and angles α, and $(\pi - \alpha)$.

Section III.—Triangles with other Data.*

135. 1°. Given s, A, B, C, solve the triangle.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a+b+c}{\sin A + \sin B + \sin C}$$

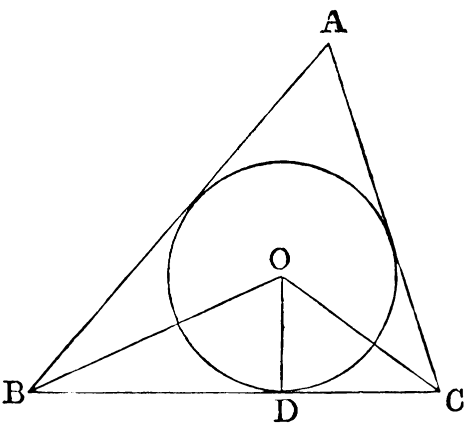
$$= \frac{s}{2\cos\frac{1}{2}A\,\cos\frac{1}{2}B\,\cos\frac{1}{2}C};$$

$$\therefore\quad a = \frac{s\sin\frac{1}{2}A}{\cos\frac{1}{2}B\,\cos\frac{1}{2}C},\quad b = \frac{s\sin\frac{1}{2}B}{\cos\frac{1}{2}C\,\cos\frac{1}{2}A},\quad c = \frac{s\sin\frac{1}{2}C}{\cos\frac{1}{2}A\,\cos\frac{1}{2}B}.$$

Cor.— $$S = \tfrac{1}{2}ab\sin C = s^2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

2°. Given S, A, B, C, solve the triangle.

$$S = \frac{a^2\sin B\sin C}{2\sin A};\qquad \therefore\ a = \sqrt{\frac{2\sin A\,.\,S}{\sin B\sin C}}.$$



* We develop some examples in the text to make the reader acquainted with the principal artifices.

3°. Given r, A, B, C, solve the triangle.

$$a = BD + DC = r(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C) = \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}},$$

$$b = \frac{r\cos\frac{B}{2}}{\sin\frac{C}{2}\cdot\sin\frac{A}{2}}, \quad c = \frac{r\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}}.$$

Cor.— $$S = \tfrac{1}{2}ab\sin C = r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

4°. Given a, A, $b + c$, solve the triangle.

$$a : b + c :: \sin\frac{A}{2} : \cos\frac{B-C}{2}.$$

Hence $B - C$ is given, &c.

Or thus: $$a^2 = (b-c)^2\cos^2\frac{A}{2} + (b+c)^2\sin^2\frac{A}{2},$$

when $(b - c)$ is given, &c.

5°. Given a, h, $b + c$, solve the triangle.

$$ah = bc\sin A,$$

and

$$a^2 = (b+c)^2 - 4bc\sin^2\frac{A}{2} = (b+c)^2 - 2ah\tan\frac{A}{2}.$$

Hence A is given.

Or thus: $ah = \frac{1}{2}\sqrt{(a+b+c)(b+c-a)[a^2-(b-c)^2]}$,

whence $b - c$ is known.

6°. Given a, h, $B - C$, solve the triangle.

$$a = h(\cot B + \cot C) = \frac{h\sin A}{\sin B\sin C} = \frac{2h\sin A}{\cos(B-C)+\cos A};$$

$$\therefore \quad \cos(B-C) = \frac{2h}{a}\sin A - \cos A \quad (\text{if } 2h/a = \cot\phi)$$

$$= \cot\phi\sin A - \cos A = \sin(A-\phi)\operatorname{cosec}\phi.$$

7°. Given A, $a + b = \beta$, $(a + c) = \gamma$, solve the triangle.

$$a^2 = (\beta - a)^2 + (\gamma - a)^2 - 2(\beta - a)(\gamma - a)\cos A;$$

$$\therefore \quad a^2(1 - 2\cos A) - 2a(\beta + \gamma)(1 - \cos A) + \beta^2 + \gamma^2 - 2\beta\gamma\cos A = 0;$$

therefore a is known.

Or thus:

$$\frac{\beta}{\gamma} = \frac{\sin A + \sin B}{\sin A + \sin C};$$

$$\therefore \quad \frac{\beta + \gamma}{\beta - \gamma} = \frac{2\sin A + 2\cos\dfrac{A}{2}\,.\cos\dfrac{B - C}{2}}{2\sin\dfrac{B - C}{2}\cos\dfrac{A}{2}}$$

$$= \frac{2\sin\dfrac{A}{2} + \cos\dfrac{B - C}{2}}{\sin\dfrac{B - C}{2}},$$

an equation of the form

$$l\sin\frac{B - C}{2} + m\cos\frac{B - C}{2} = n.$$

Hence $B - C$ is given.

8°. Given A, h, $(b - c)$, solve the triangle.

$$b - c = \frac{h}{\sin C} - \frac{h}{\sin B};$$

$$\therefore \quad \frac{b - c}{h} = \frac{\sin B - \sin C}{\sin B\,.\sin C} = \frac{4\sin\frac{1}{2}(B - C)\cos\frac{1}{2}(B + C)}{\cos(B - C) - \cos(B + C)},$$

$$\frac{b - c}{h} = \frac{2\sin\frac{1}{2}(B - C)\sin\frac{1}{2}A}{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}(B - C)}.$$

Hence $(B - C)$ is known.

EXERCISES.—XXXI.

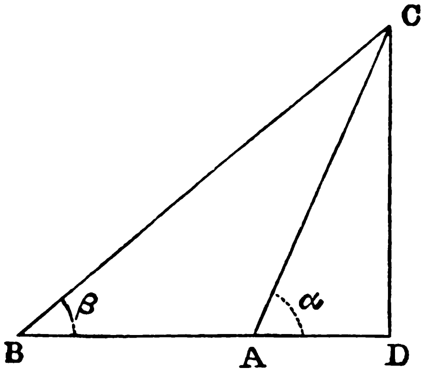
Solve the triangles from the following data :—

1. Given a, h, $b - c$.
2. ,, A, B, $a + b$.
3. ,, A, B, $s - a$.
4. ,, A, $a + b$, $a - c$.
5. ,, a, b, R.
6. ,, β(bisector of A), A, B.
7. ,, m (median), A, B.
8. ,, a, r, R.
9. ,, a, A, $\dfrac{b-c}{a}$.
10. ,, a, A, $\dfrac{b}{c}$.
11. ,, a, A, bc.
12. ,, h, h', h'' (3 altitudes).
13. Given a, A, r.
14. ,, a, A, r'.
15. ,, a, $b + c$, r.
16. ,, s, r, A.
17. ,, R, r, A.
18. ,, a, m, $B - C$.
19. ,, s, R, A.
20. ,, a, A, $b - c + h$.
21. ,, s, S, A.
22. ,, c, $A - B$, a/b.
23. ,, c, S, $\tan \frac{1}{2}A \tan \frac{1}{2}B$.
24. ,, a, S, $(B - C)$.
25. ,, A, β, m.
26. ,, r', r'', r'''.

SECTION IV.—TOPOGRAPHIC APPLICATIONS.

136. *To find the height of an inaccessible object on a horizontal plane.*

Let CD be the object; A, B, two points in the plane acces-



sible to each other, and from each of which the summit of the object can be observed.

1°. Suppose the points A, B, D to be collinear, and let the angles of elevation DAC, DBC be denoted by α, β; then in the triangle ABC the angle $ACB = \alpha - \beta$.

Hence $$AC : AB :: \sin\beta : \sin\alpha - \beta;$$

$$\therefore \quad AC = \frac{AB \sin\beta}{\sin(\alpha - \beta)}.$$

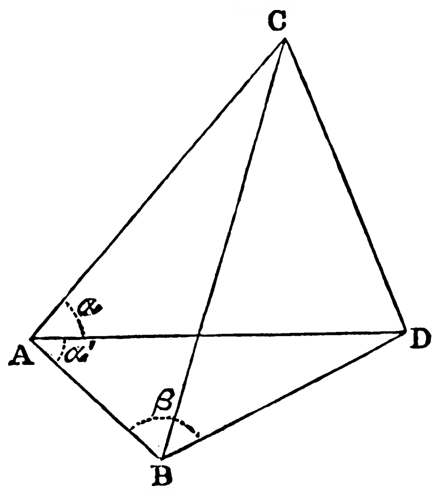
But from the right-angled triangle ADC we have

$$CD = AC \sin\alpha, \quad AD = AC \cos\alpha.$$

Hence $$CD = \frac{AB . \sin\alpha . \sin\beta}{\sin(\alpha - \beta)}, \tag{418}$$

and $$AD = \frac{AB \cos\alpha \sin B}{\sin(\alpha - \beta)}. \tag{419}$$

2°. Suppose the points A, B, D not collinear. Let the angle of elevation at A be denoted by α, and the angles at A and B



of the triangle ABD by α', β, respectively; then in the triangle ABD we have

$$AD : AB :: \sin\beta : \sin D, \quad \text{or} \quad \sin(\alpha' + \beta).$$

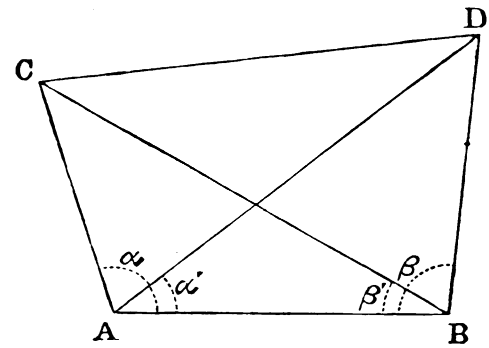
Hence $$AD = \frac{AB \sin\beta}{\sin(\alpha' + \beta)}.$$

But $$CD = AD \tan \alpha;$$

$$\therefore \quad CD = \frac{AB \tan \alpha \,.\, \sin \beta}{\sin(\alpha' + \beta)}. \tag{420}$$

137. *To find the distance between two inaccessible objects on a horizontal plane.*

Let C, D be the objects, AB the base line; then, measuring with a surveying instrument the angles marked α, α'; β, β' in



the diagram, we have, in the triangle ABC,

$$BC : AB :: \sin \alpha : \sin C, \quad \text{or} \quad \sin(\alpha + \beta').$$

Hence $$BC = \frac{AB \sin \alpha}{\sin(\alpha + \beta')}.$$

In like manner, from the triangle ABD, we have

$$BD = \frac{AB \sin \alpha'}{\sin(\alpha' + \beta)}.$$

These equations give us the sides BC, BD of the triangle BCD, and the contained angle is $(\beta - \beta')$. Hence CD can be found by Case III. of solution of triangle. In order to complete the solution, let C, D denote the angles BCD, BDC. We have

$$C + D = \pi - (\beta - \beta'),$$

$$\tan \tfrac{1}{2}(D - C) = \frac{BC - BD}{BC + BD} \tan \tfrac{1}{2}(C + D);$$

and, since the auxiliaries BC, BD are determined by their logarithms, we write

$$\frac{BC - BD}{BC + BD} = \frac{1 - \dfrac{BD}{BC}}{1 + \dfrac{BD}{BC}} = \frac{1 - \tan\phi}{1 + \tan\phi} = \tan(45^\circ - \phi).$$

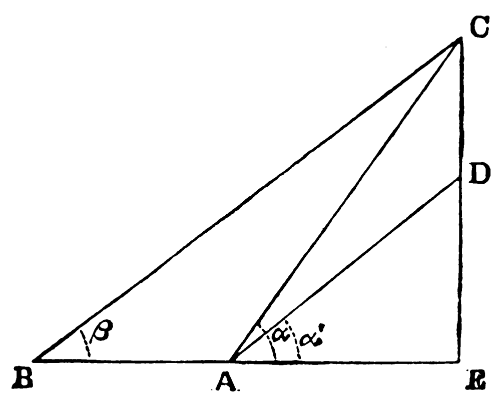
The angle ϕ is determined by the equation

$$\tan\phi = \frac{\sin\alpha' . \sin(\alpha + \beta')}{\sin\alpha . \sin(\alpha' + \beta)}, \qquad (421)$$

and $$\tan\tfrac{1}{2}(D - C) = \tan(45^\circ - \phi)\tan\tfrac{1}{2}(C + D). \qquad (422)$$

Lastly, $$CD = \frac{CB . \sin(\beta - \beta')}{\sin D}. \qquad (423)$$

138. *To find the height of an inaccessible object situated above a horizontal plane, and its height above the plane.*



Let CD be the object, AB the base line, α' the elevation of D from A. Then, from (419), we have

$$AE = \frac{AB\cos\alpha\sin\beta}{\sin(\alpha - \beta)}.$$

Hence $$DE = \frac{AB\cos\alpha\sin\beta\tan\alpha'}{\sin(\alpha - \beta)},$$

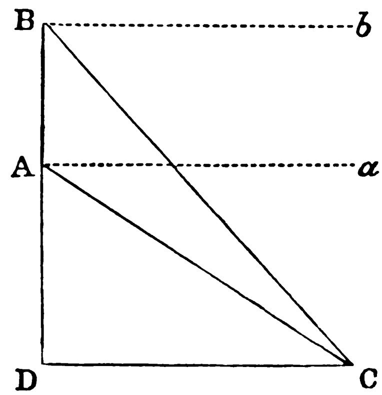
and, from (418), we have

$$EC = \frac{AB \sin \alpha \sin \beta}{\sin (\alpha - \beta)}.$$

Hence $$DC = \frac{AB \sin (\alpha - \alpha') \sin \beta}{\cos \alpha' \sin (\alpha - \beta)}. \quad (424)$$

139. *To find the distance of an object on a horizontal plane from observations made at two points in the same vertical above the plane.*

Let A, B be the places of observation, C the point observed, whose horizontal distance CD and vertical distance AD are



required. Through A, B draw the horizontal lines Aa, Bb. The angles aAC, bBC are called the angles of depression of C; let them be denoted by δ, δ'. Now, it is evident that

$$BD = CD \tan \delta', \quad AD = CD \tan \delta.$$

Hence $$AB = CD (\tan \delta' - \tan \delta);$$

$$\therefore \quad CD = \frac{AB \cos \delta \cos \delta'}{\sin (\delta - \delta')}, \quad (425)$$

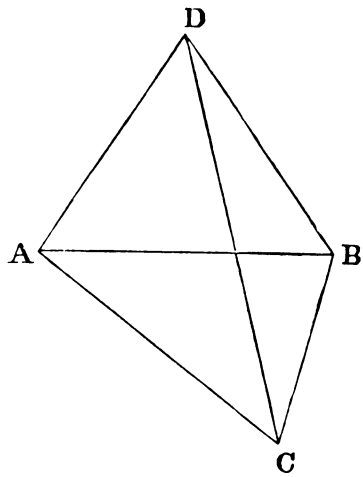
and $$AD = \frac{AB \sin \delta \cos \delta'}{\sin (\delta - \delta')}. \quad (426)$$

The same result may be obtained by conceiving the figure turned round until AB becomes horizontal; then we have the same case as § 136, only that δ, δ' are the complements of α, β.

140. *A, B, C are the angular points of a triangle, the lengths of whose sides are known; D is a point in the plane of the triangle, at which the sides AC, BC subtend given angles α, β. It is required to find the distances AD, BD.*

This is the problem of POTHENOT, or of SNELLIUS—the former French, the latter German.

Let the angles CAD, CBD be denoted by x, y, respectively; then, since the sum of the angles of the quadrilateral $ACDB$ is



four right angles, and the sum of the angles ADB, ACB is $\alpha + \beta + C$, the sum of the angles x, y is given. Now, denoting the sides of the given triangle by a, b, c, we have, from the triangles ADC, BDC,

$$CD = \frac{b \sin x}{\sin \alpha} = \frac{a \sin y}{\sin \beta}; \quad \therefore \quad \frac{\sin x}{\sin y} = \frac{a \sin \alpha}{b \sin \beta}.$$

Hence, if ϕ be an auxiliary angle, such that

$$\tan \phi = \frac{a \sin \alpha}{b \sin \beta},$$

we have

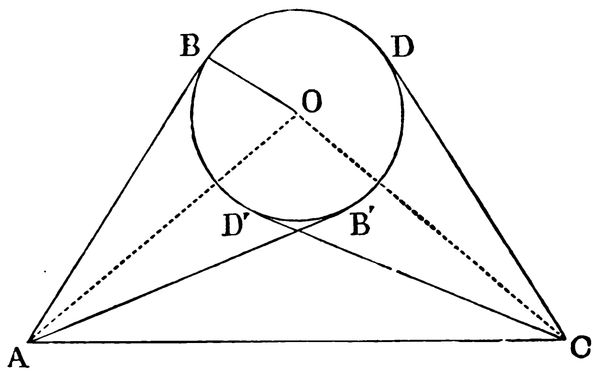
$$\frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \phi - 1}{\tan \phi + 1} = \tan(\phi - 45^\circ).$$

Hence $$\tan \tfrac{1}{2}(x - y) = \tan \tfrac{1}{2}(x + y) \tan(\phi - 45^\circ). \qquad (427)$$

Hence $x - y$ is determined; and therefore x and y are found, and the question is solved.

141. *To find the radius of an inaccessible tower.*

Draw $AC = d$, and let the angles BAC, $B'AC$, ACD, ACD'



be denoted by α, α', γ, γ', respectively; then we have

$$\frac{OA}{AC} = \frac{\sin\frac{1}{2}(\gamma + \gamma')}{\sin\frac{1}{2}(\alpha + \alpha' + \gamma + \gamma')}, \quad OB = OA \sin\tfrac{1}{2}(\alpha - \alpha').$$

Hence $$r = \frac{d \sin\frac{1}{2}(\alpha - \alpha') \sin\frac{1}{2}(\gamma + \gamma')}{\sin\frac{1}{2}(\alpha + \alpha' + \gamma + \gamma')}. \tag{428}$$

Exercises XXXII.

1. At 360 feet from the foot of a tower the elevation is half what it is at the distance of 135 feet; find the elevation.

2. At two points A, B an object DE in the vertical line CE subtends the same angle α. If AC, BC be in the same line, and equal to a, b, respectively, prove $DE = (a + b) \tan \alpha$.

3. A spherical balloon, whose diameter is δ, subtends an angle α, when the elevation of its centre is β; prove height $= \frac{1}{2}\delta \sin\beta \operatorname{cosec} \frac{1}{2}\alpha$.

4. Find the angle which a flagstaff 5 yards long, standing on a tower 200 yards high, subtends in the horizontal plane at a point 1000 feet from the foot of the tower.

5. A line AB subtends a right angle at the foot of a tower, and the angles of elevation at A and B are 30° and 18°; prove height

$$= \frac{AB}{\sqrt{2(1 + \sqrt{5})}}.$$

6. From a station A, at the foot of an inclined plane AB, the angle of elevation of the summit C of a mountain is 60°. If the inclination of AB be 30° and the angle ABC 135°, prove that height of C

$$= \frac{AB(3+\sqrt{3})}{2}.$$

7. If the angles of elevation of a balloon at three collinear stations A, B, C be α, β, γ, respectively; and if BC, CA, AB be a, b, c; prove that altitude of balloon

$$= \sqrt{\frac{abc}{a\cot^2\alpha - b\cot^2\beta + c\cot^2\gamma}}.$$

8. The heights of two poles are a, b, respectively, and their distance asunder is a. If E be a point at which the poles subtend equal angles, and if the line joining E to the foot of the shorter be perpendicular to the line between their feet, and at the distance δ from the remote pole; prove

$$\frac{1}{\delta^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

9. ABC is a triangle, right-angled at C; if the angles of a steeple at A from B and C be 45°, 15° respectively; prove

$$\tan\beta = 2\{3^{\frac{1}{4}} - 3^{-\frac{1}{4}}\}.$$

10. The angles of elevation of an object at three horizontal stations A, B, C, lying in a vertical plane passing through it, are as 1 : 2 : 3; if $AB = a$, $BC = b$, prove that altitude

$$= \frac{a}{2b}\{(a+b)(3b-a)\}^{\frac{1}{2}}.$$

11. Two stations due south of a tower, which leans towards the north with an inclination θ, are at distances a, b, respectively, from its foot; if α, β be the elevations at the stations, prove

$$b(\cot\theta - \cot\alpha) = a(\cot\theta - \cot\beta).$$

12. If the angle of elevation of a cloud from a point h feet above a lake be α, and the depression of its reflexion in the lake β, prove that its height

$$= \frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

13. Two stations a feet asunder are in the same line with the foot of a steeple, and the angles of elevation are complements; from a third station b feet nearer still the angle of elevation is double that at the first; prove altitude

$$= \tfrac{1}{2}\sqrt{(a+b)^2 - a^2}.$$

14. From a point C in a road two objects subtend the maximum angle, which is ϕ, and from a point D they appear in a right line, making an angle θ with the road; prove distance between them is $2CD \tan \frac{1}{2}(\theta + \phi)$.

15. The elevation of a balloon was observed to be 30°, and its bearing *NE*. From a second station one mile due south of the former the bearing was *NbE*; find its height.

16. The bearings of two objects A, B in the same longitude from a ship at D are *NNE* and *NEbE*; the distance AD is 10,000 metres; find BD.

17. The elevation of a tower due N of a station at A is α, and at a station B due west of A is β; prove altitude

$$= \frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}.$$

18. The shadows of two walls whose heights are h, h' are, when the sun is on the meridian, b, b' feet broad; if the walls be at right angles to each other, find the sun's altitude, and their inclinations to the meridian.

19. Two objects A, B were observed from a ship at the same instant to be in a line, bearing $N\,15°\,E$; the ship then sailed three miles *NW*, when it was found that A bore E and B *NE*; find the distance AB.

20. Two ships are sailing uniformly with velocities u, v along lines inclined at an angle α; given a, b, their distances at the same moment from the point of intersection of the lines, find their least distance asunder.

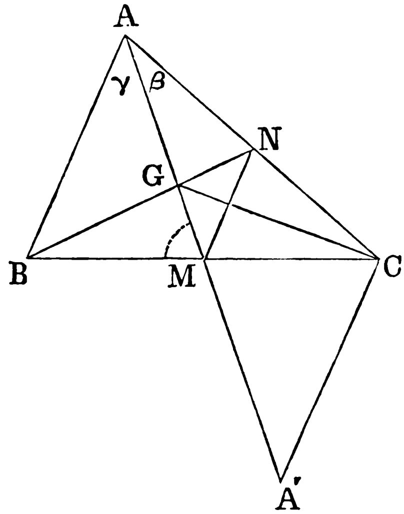
21. Given the distances between three stations A, B, C in a straight line with a tower standing on a horizontal plane, and that its elevations at these points are θ, $90° - \theta$, and 2θ, respectively. If $AB = a$, $BC = b$, prove that

$$\cos 2\theta = \frac{a}{2(a+b)}.$$

SECTION V.—MISCELLANEOUS PROPOSITIONS.

142. Properties of Medians.

Let the median AM be denoted by m, and the others by m', m''. Let the angles which m makes with AC, AB be β, γ.



Produce AM until $MA' = AM$. Draw MN parallel to AB. Join BN, cutting AM in G. Join CG.

1°. We have, in the triangle $AA'C$,

$$4m^2 = b^2 + c^2 + 2bc \cos A. \qquad (429)$$

2°. $$\frac{\sin\gamma}{\sin\beta} = \frac{b}{c} = \frac{\sin B}{\sin C}; \quad \therefore \quad \frac{\sin\gamma + \sin\beta}{\sin\gamma - \sin\beta} = \frac{\sin B + \sin C}{\sin\beta - \sin C}.$$

Hence $$\tan\tfrac{1}{2}(\gamma - \beta) = \tan^2\tfrac{1}{2}A \,.\, \tan\tfrac{1}{2}(B - C). \qquad (430)$$

3°. $$b^2 = AM^2 + MC^2 + 2AM \,.\, MC \cos AMB,$$

$$c^2 = AM^2 + MB^2 - 2AM \,.\, MB \cos AMB;$$

$$\therefore \quad b^2 - c^2 = 4AM \,.\, MC \cos AMB = 2am \cos(\widehat{a\,m}). \qquad (431)$$

4°. If h denote the perpendicular on BC, we have

$$BM = h(\cot B + \cot M), \text{ and } MC = h(\cot C - \cot M);$$

$$\therefore \quad \cot B + \cot M = \cot C - \cot M;$$

$$\therefore \quad \cot M = \frac{\cot C - \cot B}{2} \qquad (432)$$

5°. MN being the median of the triangle MAC, we have

$$\cot MNC = \cot A = \tfrac{1}{2}(\cot\beta - \cot C);$$

$$\therefore \quad \cot\beta = 2\cot A + \cot C.$$

Similarly, $$\cot\gamma = 2\cot A + \cot B. \qquad (433)$$

6°. Applying this to the triangle BGC, we have

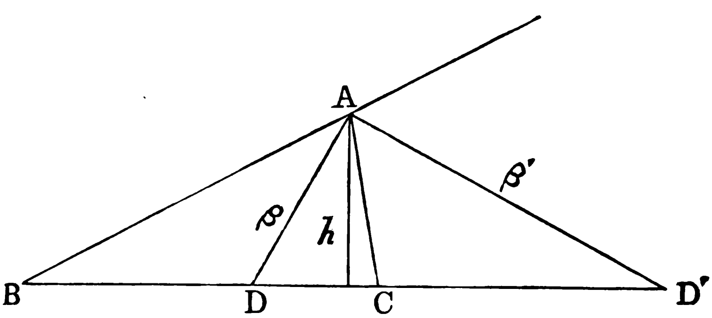
$$\cot BGM = 2\cot BGC + \cot GBC = 2\cot BGC + 2\cot B + \cot C.$$

Hence
$$\begin{aligned} -\cot(m, m') &= 2\cot(m', m'') + 2\cot B + \cot C \\ -\cot(m', m'') &= 2\cot(m'', m) + 2\cot C + \cot A \\ -\cot(m'', m) &= 2\cot(m, m') + 2\cot A + \cot B. \end{aligned}$$

Hence $$-\cot(m, m') = \tfrac{1}{3}(2\cot A + 2\cot B - \cot C). \qquad (434)$$

143. Bisectors of Angles.

Let AD be the internal, AD' the external bisector of A.



We have $$S = ABD + ADC;$$

$$\therefore \quad \tfrac{1}{2}bc\sin A = \tfrac{1}{2}c\beta\sin\frac{A}{2} + \tfrac{1}{2}b\beta\sin\frac{A}{2}.$$

Hence $$\beta = \frac{2bc\cos\frac{A}{2}}{b+c}.$$

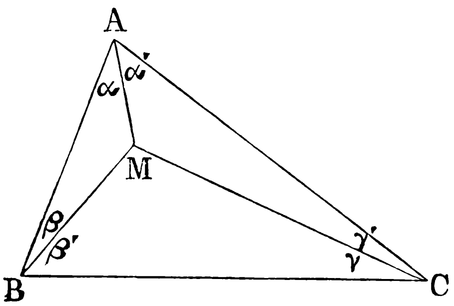
Similarly, $$\beta' = \frac{2bc\sin\frac{A}{2}}{b-c}. \qquad (435)$$

We have also

$$h = \beta\sin ADB = \beta\cos\tfrac{1}{2}(B-C) = \beta'\sin\tfrac{1}{2}(B-C). \qquad (436)$$

144. Concurrent Lines.

1°. If lines drawn from a point M to the summits of a triangle



divide its angles into parts α, α'; β, β'; γ, γ', respectively,

$$\sin\alpha\,\sin\beta\,\sin\gamma = \sin\alpha'\,\sin\beta'\,\sin\gamma'. \tag{437}$$

And, reciprocally, if this relation holds, the lines are concurrent. This follows from the proposition that the sides of a triangle are proportional to the sines of their opposite angles.

2°. If in equation (437) the angles α, β, γ be equal; then, denoting each of them by ω, we get

$$\sin^3\omega = \sin(A-\omega)\,\sin(B-\omega)\,\sin(C-\omega). \tag{438}$$

The angle ω is called the *Brocard angle* of the triangle. By expanding the second side, and dividing by $\sin^3\omega \sin A \sin B \sin C$, we get

$$\cot^3\omega - \cot^2\omega\,\Sigma\cot A + \cot\omega\,\Sigma\cot A\cot B$$
$$-(\cot A\cot B\cot C + \operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C) = 0;$$

but $$\Sigma\cot A\cot B = 1,$$

and $$\cot A\cot B\cot C + \operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C = \Sigma\cot A.$$

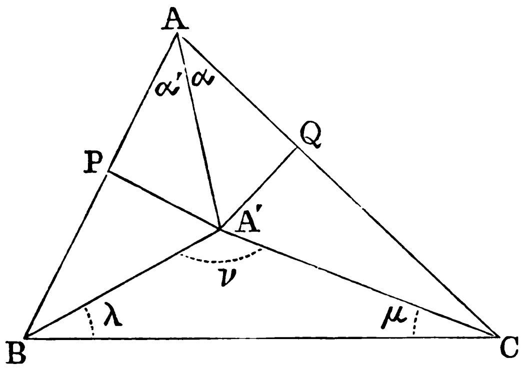
Hence $$\cot^3\omega + \cot\omega = (1+\cot^2\omega)\,\Sigma\cot A;$$

$$\therefore\quad \cot\omega = \Sigma\cot A = \cot A + \cot B + \cot C. \tag{439}$$

Hence $$1 + \cot^2\omega = \cot^2 A + \cot^2 B + \cot^2 C + 3,$$

or $$\sec^2\omega = \sec^2 A + \sec^2 B + \sec^2 C. \tag{440}$$

3°. *Three triangles directly similar are constructed on the sides of the triangles,* viz. *BCA', CAB', ABC'; it is required to find the condition that AA', BB', CC' are concurrent.*



From A' let fall the perpendiculars $A'P$, $A'Q$, and let the angles of the triangle BCA' be λ, μ, ν, respectively, and the angles into which AA' divides A be α, α'; then we have

$$\frac{\sin \alpha}{\sin \alpha'} = \frac{A'Q}{A'P} = \frac{A'C \sin (C - \mu)}{A'B \sin (B - \lambda)}$$

$$= \frac{\sin \lambda \sin (C - \mu)}{\sin \mu \sin (B - \lambda)} = \frac{\sin C (\cot \mu - \cot C)}{\sin B (\cot \lambda - \cot B)}.$$

Hence, from (437), the condition of concurrence is

$$(\cot \lambda - \cot A)(\cot \lambda - \cot B)(\cot \lambda - \cot C)$$
$$= (\cot \mu - \cot A)(\cot \mu - \cot B)(\cot \mu - \cot C),$$

or $$(\cot^3 \lambda - \cot^3 \mu) - (\cot^2 \lambda - \cot^2 \mu) \Sigma \cot A$$
$$+ (\cot \lambda - \cot \mu) \Sigma \cot A \cot B = 0.$$

Hence $$\cot \lambda - \cot \mu = 0,$$

or $$\cot^2 \lambda + \cot \lambda \cot \mu + \cot^2 \mu - (\cot \lambda + \cot \mu) \cot \omega + 1 = 0,$$

ω being the Brocard angle (2°.)

We have also

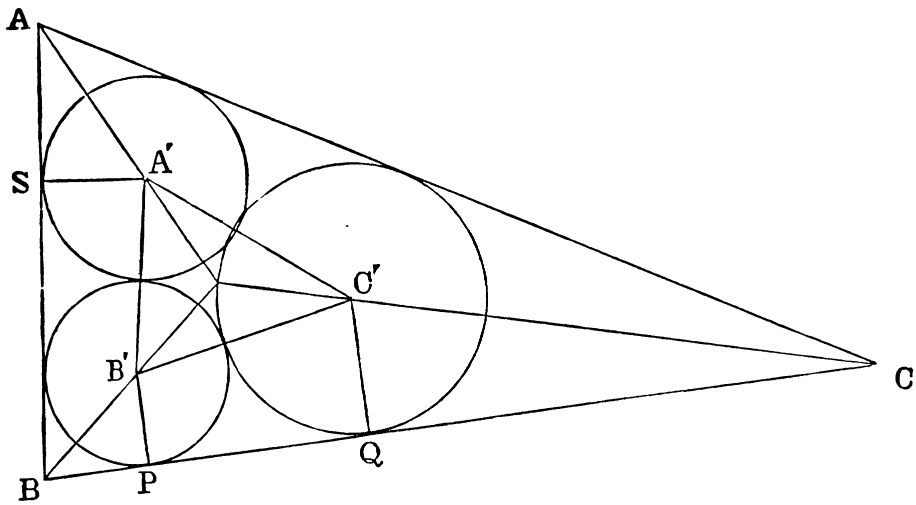
$$\cot \lambda \cot \mu + \cot \mu \cot \nu + \cot \nu \cot \lambda - 1 = 0.$$

Hence, by addition to the preceding, we get

$$\cot \lambda + \cot \mu + \cot \nu = \cot \omega. \quad \text{(Neuberg.)} \qquad (441)$$

145. Malfatti's Problem.

To inscribe in a triangle three circles in mutual contact, and each touching two sides of the triangle.



Let the lines AS, BP, CQ be denoted by x, y, z; then we have

$$a = BP + QC + PQ = y + z + 2\sqrt{yz \tan\frac{B}{2} \tan\frac{C}{2}}.$$

Now $$\tan\frac{B}{2}\tan\frac{C}{2} = \frac{s-a}{s} = 1 - \frac{a}{s};$$

and putting

$$a/s = \sin^2\alpha, \quad x/s = \cos^2\lambda, \quad y/s = \cos^2\mu, \quad z/s = \cos^2\nu,$$

we have $$\sin^2\alpha = \cos^2\mu + \cos^2\nu + 2\cos\mu\cos\nu\cos\alpha;$$

$$\therefore \quad \mu + \nu + \alpha = \pi.$$

Similarly, putt n $/s = \sin^2\beta, \quad c/s = \sin^2\gamma,$

we get $$\nu + \lambda + \beta = \pi, \text{ and } \lambda + \mu + \gamma = \pi.$$

Hence, putting $$\alpha + \beta + \gamma = 2\sigma,$$

we have $$\lambda + \sigma - \alpha = \frac{\pi}{2};$$

$$\therefore \quad \cos\lambda = \sin(\sigma - \alpha);$$

therefore

$$x = s\sin^2(\sigma - \alpha), \quad y = s\sin^2(\sigma - \beta), \quad z = s\sin^2(\sigma - \gamma).$$

This elegant solution is due to Lechmütz, *Nouv Annales*, tom. v. p. 61. For a purely Geometrical solution, see "Sequel to Euclid," p. 154.

146. To find the Radii of Malfatti's Circles.

If we denote the radii by ρ_1, ρ_2, ρ_3, respectively, the equation $BP + PQ + QC = a$ may be written

$$\rho_2 \cot \tfrac{1}{2} B + 2\sqrt{\rho_2 \rho_3} + \rho_3 \cot \tfrac{1}{2} C = r(\cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C);$$

or putting $\rho_1 = rx, \quad \rho_2 = ry, \quad \rho_3 = rz,$

we get

$$y \cot \tfrac{1}{2} B + 2\sqrt{yz} + z \cot \tfrac{1}{2} C = \cos \tfrac{1}{2} A/(\sin \tfrac{1}{2} B \sin \tfrac{1}{2} C).$$

Similarly,

$$z \cot \tfrac{1}{2} C + 2\sqrt{zx} + x \cot \tfrac{1}{2} A = \cos \tfrac{1}{2} B/(\sin \tfrac{1}{2} C \sin \tfrac{1}{2} A). \quad (1)$$

Hence, eliminating the absolute term, we get

$$y \cos^2 \tfrac{1}{2} B \sin \tfrac{1}{2} C + 2 \sin \tfrac{1}{2} B \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C \sqrt{yz}$$
$$+ z \cos \tfrac{1}{2} C (\sin \tfrac{1}{2} B \cos \tfrac{1}{2} B - \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A)$$
$$= 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A \sin \tfrac{1}{2} C \sqrt{zx} + x \cos^2 \tfrac{1}{2} A \sin \tfrac{1}{2} C;$$

but

$$\cos \tfrac{1}{2} C (\sin \tfrac{1}{2} B \cos \tfrac{1}{2} B - \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A) = \sin \tfrac{1}{2} C (\sin^2 \tfrac{1}{2} B - \sin^2 \tfrac{1}{2} A).$$

Hence, by an easy reduction,

$$\sqrt{y} \cos \tfrac{1}{2} B + \sqrt{z} \sin \tfrac{1}{2} B = \sqrt{x} \cos \tfrac{1}{2} A + \sqrt{z} \sin \tfrac{1}{2} A.$$

Similarly,

$$\sqrt{z} \cos \tfrac{1}{2} C + \sqrt{x} \sin \tfrac{1}{2} C = \sqrt{y} \cos \tfrac{1}{2} B + \sqrt{x} \sin \tfrac{1}{2} B;$$

$$\therefore \quad \sqrt{z} (\cos \tfrac{1}{2} C + \sin \tfrac{1}{2} B - \sin \tfrac{1}{2} A)$$
$$= \sqrt{x} (\cos \tfrac{1}{2} A + \sin \tfrac{1}{2} B - \sin \tfrac{1}{2} C); \quad (2)$$

N 2

$$\therefore \quad \sqrt{\frac{z}{x}} = \frac{1 + \tan\frac{1}{4}A}{1 + \tan\frac{1}{4}C};$$

and from (1) we have

$$z \cot\tfrac{1}{2}C + 2\sqrt{zx} + x\cos\tfrac{1}{2}A = \cot\tfrac{1}{2}C + \cot\tfrac{1}{2}A.$$

Hence, eliminating z, we get

$$x \text{ or } \frac{\rho_1}{r} = \tfrac{1}{2}(1 + \tan\tfrac{1}{4}B)(1 + \tan\tfrac{1}{4}C)/(1 + \tan\tfrac{1}{4}A);$$

$$\therefore \quad \rho_1 = \frac{r}{2}(1 + \tan\tfrac{1}{4}B)(1 + \tan\tfrac{1}{4}C)/(1 + \tan\tfrac{1}{4}A). \qquad (442)$$

Cor.—$\rho_1\rho_2\rho_3 = \dfrac{r^3}{8}(1 + \tan\tfrac{1}{4}A)(1 + \tan\tfrac{1}{4}B)(1 + \tan\tfrac{1}{4}C). \qquad (443)$

This solution, slightly altered, is taken from Hymers' *Trigonometry* (4th ed.), 1858. A method nearly identical is given by Catalan in the Bulletin of the *Acad. Royale Belgique*, 1874.

147. Maximum and Minimum in Trigonometry.

1°. *To divide an angle α not greater than π into positive parts, so that the product of their sines may be a maximum.*

If x, y be the parts, the conditions of the problem give us

$$x + y = \alpha, \text{ and } 2\sin x \sin y \text{ a maximum.}$$

Hence $\quad \cos(x - y) - \cos(x + y)$ is a maximum;

or $\quad \cos(x - y) - \cos\alpha$ is a maximum.

Hence $\quad x = y.$

Cor.—If $x + y + z = \alpha$, and $\sin x \sin y \sin z$, a maximum, $x = y = z$.

For if x be unequal to y, leaving z unaltered, we can replace $\sin x \sin y$ by $\sin\frac{1}{2}(x + y)\sin\frac{1}{2}(x + y)$, and we get a greater product.

Example.—The maximum triangle inscribed in a given circle is equilateral.

For $S = 2R^2 \sin A \sin B \sin C, \quad A + B + C = \pi.$

Hence $$A = B = C.$$

2°. *Divide a given angle not greater than π into two parts, the sum of whose sines is a maximum.*

Here $x + y = a$, and $\sin x + \sin y$ a maximum.

Hence $2 \sin \frac{1}{2} a \cos \frac{1}{2}(x - y)$ is a maximum;

$$\therefore \quad x = y.$$

Cor.—If $x + y + z = a$, and $\sin x + \sin y + \sin z$ a maximum, $x = y = z$.

3°. *Divide a given angle not greater than $\frac{\pi}{2}$ into two parts, the product of whose tangents is a maximum.* The parts must be equal.

Cor.—If $x + y + z = a$, and $\tan x \tan y \tan z$ a maximum, $x = y = z$.

Application.—*Construct a maximum triangle with a given perimeter*

$$S = s^2 \tan \frac{A}{2} . \tan \frac{B}{2} \tan \frac{C}{2};$$

$$\therefore \quad A = B = C.$$

4°. *To find x, so that $a \sin x + b \cos x$ may be a maximum.*

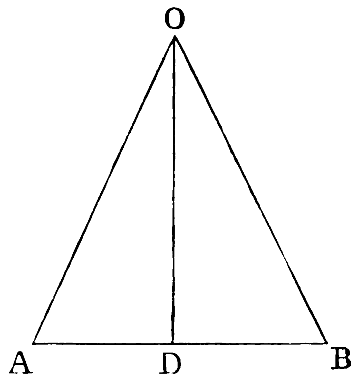
$$a \sin x + b \cos x = a \left(\sin x + \frac{b}{a} \cos x\right)$$

$$= a (\sin x + \tan \phi \cos x), \text{ if } b/a = \tan \phi;$$

$$\therefore \quad a \sin x + b \cos x = a \sin (x + \phi)/\cos \phi;$$

$$\therefore \quad x + \phi = \frac{\pi}{2},$$

148. *Radii of incircles and circumcircles of regular polygons. Being given the side of a regular polygon of n sides, it is required to find the radii of its circumcircle and incircle.*



Let AB, a side of the polygon, be denoted by a. Let O be the common centre of the circles, R, r their radii. Draw OD perpendicular to AB, then the angle

$$AOD = \pi/n;$$

and from the triangle AOD we have

$$AO \,.\, \sin AOD = AD; \text{ that is, } R \sin \pi/n = a/2;$$

$$\therefore \quad R = \frac{a}{2 \sin \pi/n}. \tag{444}$$

Again,

$$AD/OD = \tan AOD; \text{ that is, } a/2r = \tan \pi/n;$$

$$\therefore \quad r = \tfrac{1}{2} a \cot \pi/n. \tag{445}$$

Cor. 1.—The area of the polygon, being n times the triangle AOB, is

$$= n \,.\, \frac{a}{2} \,.\, r = \frac{na^2}{4} \cot \frac{\pi}{n}. \tag{446}$$

Cor. 2—Area $= \dfrac{nR^2}{2} \sin^2 \dfrac{\pi}{n} = nr^2 \tan^2 \dfrac{\pi}{n}$. (447)

Cor. 3.—The area of a circle whose radius is r is πr^2. (448)

For the circle may be regarded as the limit of a circumscribed polygon, when the number of sides increases without limit. Now, when n is indefinitely great,

$$\tan \pi/n = \pi/n.$$

Hence, from *Cor.* 2, we get area $= \pi r^2$.

Cor. 4.—If θ be the circular measure of the angle of a sector, the area

$$= \theta r^2/2. \qquad (449)$$

Cor. 5.—In a unit circle, the lengths of an arc, the circular measure of the corresponding angle, and twice the area of the sector formed by joining its extremities to the centre, are expressed numerically by the same quantity.

Cor. 6.—The circular functions of an angle may be regarded either as functions of the angle of the corresponding arc, or twice the corresponding sector.

Exercises.—XXXIII.

1. Find in functions of the angles of a triangle the ratios of the segments in which the altitude AH and the bisector BE of the angle B mutually divide each other.

2. Find in a function of S, A, B, C the area of the triangle formed by the altitude AH, and the bisectors BE, CF of the angles B, C. 2°. That formed by the bisector AD, and the altitudes BH', CH''.

3. Find the area of the orthocentre triangle.

4. Find in a function of R, A, B, C the distance of the orthocentre from the circumcentre.

5. If m, m' denote the median and the symmedian of the side a of the triangle ABC, prove

$$\frac{m + m'}{m - m'} = \left(\frac{b + c}{b - c}\right)^2.$$

6. If the ratios which the sides of a triangle bear to the corresponding altitudes be l, m, n, prove that

$$l^2 + m^2 + n^2 = 2(lm + mn + nl).$$

7. The area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the inpolygon and circumpolygon of n sides.

8. The area of a regular circumpolygon of $2n$ sides is a harmonic mean between the area of a regular circumpolygon of n sides and that of an inpolygon of $2n$ sides.

9. The radii of the circles each touching the sides b, c of the triangle ABC, and each touching the incircle, are

$$r\tan^2\tfrac{1}{4}(B + C), \quad \text{and} \quad r\cot^2\tfrac{1}{4}(B + C). \tag{450}$$

10. The radii of the circles touching the sides b, c of the triangle ABC, and each touching the circumcircle, are

$$r\sec^2\tfrac{1}{2}A, \quad \text{and} \quad r'\sec^2\tfrac{1}{2}A. \tag{451}$$

11. Prove that the angles of intersection of the nine-points circle with the sides of the triangle are—

$$\tfrac{1}{2}(B - C), \quad \tfrac{1}{2}(C - A), \quad \tfrac{1}{2}(A - B), \text{ respectively.}$$

12. The incircle of a triangle touches the sides in D, E, F, and the feet of the altitudes are D', E', F'; prove

$$DD'\sin A + EE'\sin B + FF'\sin C = 0.$$

13. Find the relation between the radii of the incircles of the four triangles ABC, EAF, FBD, DCE.

14. The altitudes of ABC meet the circumcircle in A', B', C'; find the area of the hexagon comprised between the sides of the triangles ABC, $A'B'C'$.

15. If A' be the point on the circumcircle of the triangle ABC which is diametrically opposite to A, and if the tangent in A' meet the side AB in A'', and AC in A''', find the areas of the triangles $A''B''C''$, $A'''B'''C'''$.

16. Through a point of intersection of two circles draw a line cutting both circles, so that the sum of the chords or their rectangle may be a maximum.

17. Inscribe the maximum rectangle in a sector of a circle.

18. If $x + y$ be constant, find the maximum value of $\sin^2 x + \sin^2 y$.

19. Being given two rectangular lines OX, OY, and two fixed points A, A' in OX, find a point M in OY so that the angle AMA' is a maximum.

20. Find the maximum value of $a^2 \tan x + b^2 \cot x$.

21. Find the maximum value of $\sin x + \sin y$, $\cos x + \cos y$ being constant.

22. The area of the incircle of a triangle : area of triangle : : π : $\cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C$.

23. Calculate the distance between the incentre and the centre of the nine-points circle.

24. If ρ_1, ρ_2, ρ_3 be the radii of MALFATTI's circles, prove that

$$\frac{\sqrt{\rho_1\rho_2} + \sqrt{\rho_2\rho_3} + \sqrt{\rho_3\rho_1}}{\sqrt{r'r''} + \sqrt{r''r'''} + \sqrt{r'''r'}} + \frac{\rho_1}{r'} + \frac{\rho_2}{r''} + \frac{\rho_3}{r'''} = 1.$$

25. If ρ', r'', r''' be the radii of three circles, each touching two sides of a triangle, and touching its nine-points circle, prove that

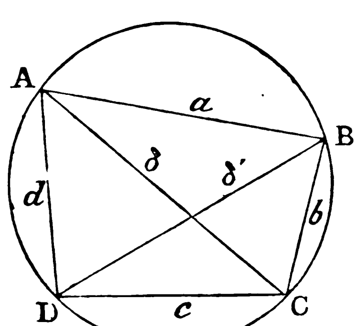
$$\frac{\rho'}{r'} + \frac{\rho''}{r''} + \frac{\rho'''}{r'''} = \frac{a^2 + b^2 + c^2}{(a + b + c)^2}.$$

26. A, B, C, D, E, F are the summits of a regular hexagon whose incentre is O. Join AC, cutting OB in P; PD, cutting OC in Q; QE, cutting OD in R, &c. Prove that OP, OQ, OR, OS, &c., are proportional to the reciprocals of the natural numbers 1, 2, 3, 4, &c.

SECTION VI.—QUADRILATERALS.

149. *To find the area of a cyclic quadrilateral in terms of its sides.*

Let $ABCD$ be the quadrilateral, and let the sides and diago-



nals be denoted as in the diagram. Then, since the angles B, D are supplemental, we have

$$\delta^2 = a^2 + b^2 - 2ab\cos B = c^2 + d^2 + 2ca\cos D.$$

Hence
$$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)};$$

$$\therefore\quad \sin^2\tfrac{1}{2}B = \frac{(c+d)^2 - (a-b)^2}{4(ab+cd)},\quad \cos^2\tfrac{1}{2}B = \frac{(a+b)^2 - (c-d)}{4(ab+cd)}.$$

Now, putting s for the semiperimeter of the quadrilateral, these equations give

$$\sin\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-b)}{ab+cd}},\quad \cos\tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-d)}{ab+cd}}.$$

Hence
$$\sin B = 2\,\frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}.$$

But, if S denote the area,
$$S = \tfrac{1}{2}(ab + cd)\sin B.$$

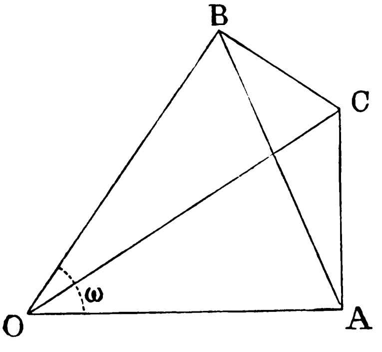
Hence
$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{452}$$

Cor.—If we substitute the value of $\cos B$ in the equation
$$\delta^2 = a^2 + b^2 - 2ab\cos B,$$

we get
$$\delta^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}. \tag{453}$$

Similarly,
$$\delta'^2 = \frac{(ac + bd)(ab + cd)}{ad + bc}. \tag{454}$$

150. Quadrilateral Birectangular.*



* This quadrilateral occurs frequently. Legendre discussed it in the early editions of his Geometry.

Let $OACB$ be a quadrilateral, in which are given the angle $AOB = \omega$, the side $OA = a$, $OB = b$, and the angles A, B right. It is required to calculate the remaining sides and diagonals.

1°. $$AB^2 = a^2 + b^2 - 2ab \cos \omega. \tag{455}$$

2°. $$OC = AB \operatorname{cosec} \omega = \sqrt{(a^2 + b^2 - 2ab \cos \omega)} \operatorname{cosec} \omega. \tag{456}$$

For OC is the diameter of the circumcircle of the triangle AOB.

3°. $$OA = OB \cos \omega + BC \sin \omega;$$

therefore $$BC = (a - b \cos \omega)/\sin \omega. \tag{457}$$

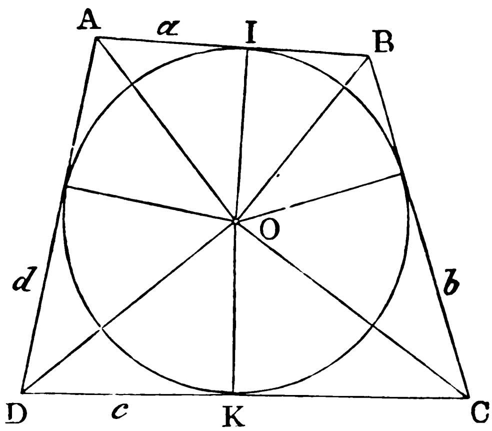
4°. Similarly, $$CA = (b - a \cos \omega)/\sin \omega. \tag{458}$$

151. Quadrilateral Circumscribed.

In order that a quadrilateral formed with four lines a, b, c, d may be circumscriptible, it is necessary that

$$a + c = b + d.$$

It suffices then to know the four segments AI, IB, CK, KD.



Denoting them by α, β, γ, δ, we have

$$\tan \frac{A}{2} = \frac{r}{\alpha}, \quad \tan \frac{B}{2} = \frac{r}{\beta}, \quad \tan \frac{C}{2} = \frac{r}{\gamma}, \quad \tan \frac{D}{2} = \frac{r}{\delta};$$

but $$\frac{A+B+C+D}{2} = \pi.$$

Hence $$\tan (A + B + C + D)/2$$

$$= \frac{\Sigma \tan \frac{A}{2} - \Sigma \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{1 - \Sigma \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{A}{2} . \tan \frac{B}{2} . \tan \frac{C}{2} . \tan \frac{D}{2}} = 0.$$

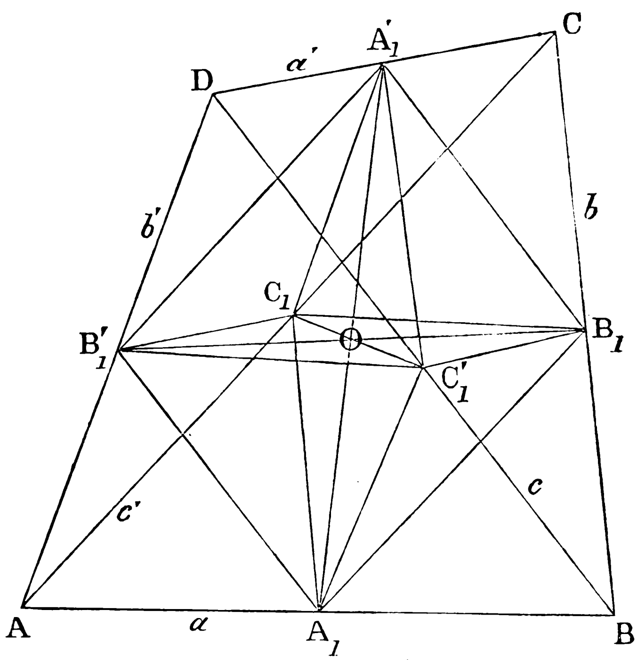
Hence $$r\Sigma \frac{1}{\alpha} = r^3 \Sigma \frac{1}{\alpha\beta\gamma};$$

$$\therefore \quad r^2 = \frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha + \beta + \gamma + \delta}. \tag{459}$$

$$S = r(\alpha + \beta + \gamma + \delta) = \sqrt{(\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)}. \tag{460}$$

152. Any Quadrilateral.

(a, a'), (b, b'), (c, c') are the pairs of opposite sides and the diagonals; A_1, A_1', B_1, B_1', C_1, C_1', the middle points of these



lines; and let the medians A_1A_1', B_1B_1', C_1C_1' be denoted by α, β, γ.

1°. The three parallelograms $A_1B_1A_1'B_1'$, $B_1C_1B_1'C_1'$, $C_1A_1C_1'A_1'$ give the following relations:—

$$c^2 + c'^2 = 2(\alpha^2 + \beta^2),\quad a^2 + a'^2 = 2(\beta^2 + \gamma^2),\quad b^2 + b'^2 = 2(\gamma^2 + \alpha^2). \tag{461}$$

Hence

$$a^2 + a'^2 - b^2 - b'^2 = 2(\beta^2 - \alpha^2),\quad b^2 + b'^2 - c^2 - c'^2 = 2(\gamma^2 - \beta^2),$$

$$c^2 + c'^2 - a^2 - a'^2 = 2(\alpha^2 - \gamma^2). \tag{462}$$

From (462), we infer also

$$a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2 = 4(\alpha^2 + \beta^2 + \gamma^2);$$

therefore

$$4\alpha^2 = b^2 + b'^2 + c^2 + c'^2 - a^2 - a'^2,\quad 4\beta^2 = c^2 + c'^2 + a^2 + a'^2 - b^2 - b'^2,$$

$$4\gamma^2 = a^2 + a'^2 + b^2 + b'^2 - c^2 - c'^2. \tag{463}$$

2°. If we apply the relation (§ 142, 3°) to the triangle A_1OB_1, we get

$$\alpha^2 - \beta^2 = cc' \cos(c, \hat{\ } c'),\quad \beta^2 - \gamma^2 = aa' \cos(a, \hat{\ } a'),$$

$$\gamma^2 - \alpha^2 = bb' \cos(b, b'). \tag{464}$$

Hence $\quad aa' \cos(a, a') + bb' \cos(b, b') + cc' \cos(c, c') = 0. \quad (465)$

From (463) and (465), we get

$$a^2 + a'^2 - b^2 - b'^2 = -2cc' \cos(c, c'). \tag{466}$$

3°. We have $\qquad S = \frac{1}{2} cc' \sin(c, c').$

Hence $\qquad 4S = (b^2 + b'^2 - a^2 - a'^2) \tan(c, c'). \quad (467)$

From $S = \frac{1}{2} cc' \sin(c, c')$, we have the formula of Breitschneider and Dostor, viz.,

$$S = \tfrac{1}{2}\{c^2c'^2(1 - \cos^2(cc'))\}^{\frac{1}{2}} = \tfrac{1}{4}\sqrt{4c^2c'^2 - (a^2 + a'^2 - b^2 - b'^2)^2}. \tag{468}$$

Cor. 1.—If the quadrilateral be cyclic, we have $cc' = aa' + bb'$, and the formula (469) becomes identical with (458).

Cor. 2.—If the quadrilateral be circumscriptible, we have $a + a' = b + b'$, and the formula (468) becomes

$$\tfrac{1}{2}\sqrt{(cc' + bb' - aa')(cc' + aa' - bb')}. \qquad (469)$$

Cor. 3.—The area of any quadrilateral can be expressed in terms of the sides and a pair of opposite angles. Thus, employing the notation of § 149, we have

$$S = \tfrac{1}{2}(ad \sin A + bc \sin C);$$

but $$\cos A = \frac{a^2 + d^2 - \delta'^2}{2ad}, \quad \cos C = \frac{b^2 + c^2 - \delta'^2}{2bc}.$$

Hence $$2ad \cos A - 2bc \cos C = a^2 + d^2 - b^2 - c^2;$$

$$\therefore \quad (a + d)^2 - (b - c)^2 = 4ad \cos^2 \tfrac{1}{2} A + 4bc \sin^2 \tfrac{1}{2} C,$$

and $$(b + c)^2 - (a - d)^2 = 4ad \sin^2 \tfrac{1}{2} A + 4bc \cos^2 \tfrac{1}{2} C.$$

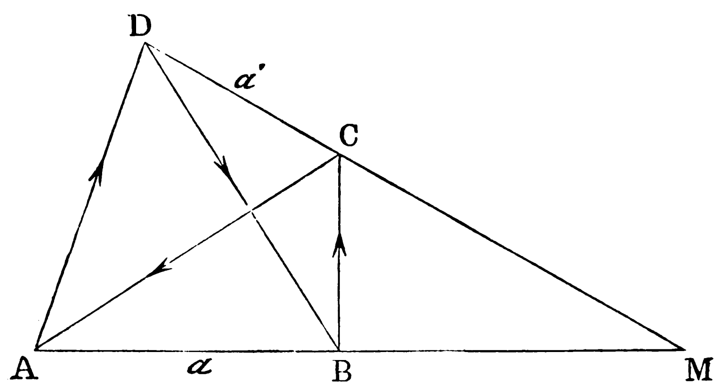
Hence, multiplying and reducing, we get

$$(s - a)(s - b)(s - c)(s - d)$$
$$= \tfrac{1}{4}(ad \sin A + bc \sin C)^2 + abcd \cos^2 \tfrac{1}{2}(A + C);$$

$$\therefore \; S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \tfrac{1}{2}(A + C).$$

Cor. 4.—If the four sides of a quadrilateral be given, it is a maximum when it is cyclic.

153. Four points A, B, C, D are the vertices of three quadrilaterals, viz., $ABCD$, $ADBC$, $ACBD$. The expressions got



for the area of $ABCD$ have analogues for the areas of the others. Thus, consider the quadrilateral $ADBC$. The area of

this is by the usual convention $= \Delta ADB - \Delta ACB$

$$= \tfrac{1}{2}\, a\,(\text{diff. of perpendiculars from } D \text{ and } C \text{ on } a)$$

$$= \tfrac{1}{2}\, aa' \sin(\widehat{aa'}) = \tfrac{1}{4}\sqrt{4a^2a'^2 = (b^2+b'^2-c^2-c'^2)^2}. \quad (470)$$

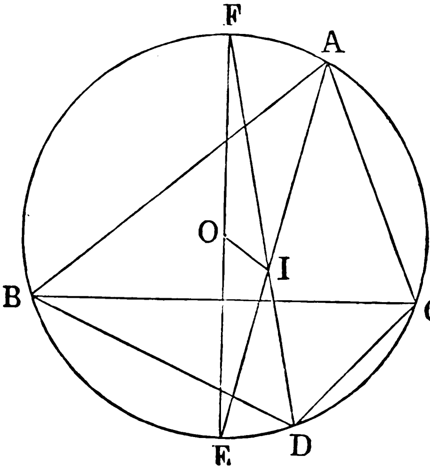
If the quadrilateral be cyclic, $aa' = cc' - bb'$, and we get

$$\Delta ADB - \Delta ACB = \text{area } ADBC$$

$$= \tfrac{1}{4}\sqrt{(c+c'+b+b')(c+c'-b-b')(c-c'+b-b')(c-c'-b+b')}.$$

(471)

154. *If a quadrilateral be inscriptible and circumscriptible, it is required to express the distance between the centres of the circles in terms of their radii.*



Let $ABCD$ be the quadrilateral, I the incentre. Join AI, DI, and produce to meet the circumcircle in E and F. Join EF, then EF is evidently the diameter of the circumcircle. Let O be the circumcentre, R, r the radii; then we have

$$r/AI = \sin\tfrac{1}{2}A, \quad r/DI = \sin\tfrac{1}{2}D = \cos\tfrac{1}{2}A.$$

Hence $$1/AI^2 + 1/DI^2 = 1/r^2.$$

Again, $$AI.IE = R^2 - \delta^2 = DI.IF.$$

Hence $$IE^1 + IF^2 = (R^2-\delta^2)^2/r^2;$$

but $$IE^2 + IF^2 = 2OE^2 + 2OI^2 = 2R^2 + 2\delta^2;$$

$$\therefore \quad 2R^2 + 2\delta^2 = (R^2 - \delta^2)^2/r^2;$$

$$\therefore \quad 1/(R+\delta)^2 + 1/(R-\delta)^2 = 1/r^2. \qquad (472)$$

The foregoing elegant demonstration is due to R. F. Davies, *Educational Times*, vol. xxxii. For a Geometrical proof, see "Sequel," p. 109, Prop. XIV.

Exercises.—XXXIV.

1. If a, b, c, d denote the sides of a quadrilateral; prove that

$$d^2 = a^2 + b^2 + c^2 - 2ab\cos(\widehat{ab}) - 2bc\cos(\widehat{bc}) - (2ca\cos(\widehat{ca}).$$

2. If a quadrilateral be both inscriptible and circumscriptible, prove that its area is equal to the square root of the product of its sides.

3. Prove that the sides and diagonals of any quadrilateral (§ 152) are connected by the relations

$$\Sigma\,(a^2a'^2)(b^2+b'^2+c^2+c'^2-a^2-a'^2) - a^2b^2c^2 - a^2b'^2c'^2 - b^2c'^2a'^2 - c^2a'^2b'^2 = 0. \qquad (474)$$

4. In a given quadrilateral $ABCD$, if B', D' be the projections of B, D on AC, and A', C' the projections of A, C on BD; if θ be the angle between the diagonals of $ABCD$, prove that area $A'B'C'D' = ABCD\cos^2\theta$.

5. Through a point P at the distance $PO = d$ from the centre of a circle, two secants PAB, PDC are drawn, making angles α, β with PO; find the area of the quadrilateral $ABCD$ as a function of R, d, α, β.

6. a, b, c, d are the sides of a cyclic quadrilateral; calculate the angle of intersection of the opposite sides a, c.

7. In the same case, find the area of the quadrilateral whose summits are the incentres of the triangles ABC, ABD, ACD, BCD, and prove that this quadrilateral is rectangular.

8. Through a point P outside a circle draw a secant PAB, such that the projections of the chord AB on the diameter may be constant.

9. Through a point P in the base BC of a triangle draw a secant, cutting the other sides in D and E, so that the projection on $BC = k$.

10. Through a point P draw a secant PAB, so that the tangents through A, B will intercept on the diameter through P a segment $= k$.

11. If each of the sides of a circumscriptible quadrilateral be given in magnitude, and one of them given in position, the locus of the incentre is a circle. (Malet.)

*12. Prove (see fig., § 151), $ab \sin^2 \frac{1}{2} B = cd \sin^2 \frac{1}{2} D$.

13. ,, $S = \sqrt{abcd} \,.\, \sin \frac{1}{2}(A + C)$.

14. If lines OP, OQ, OR, OS be drawn, meeting the sides a, b, c, d in the points P, Q, R, S, respectively, and making the angles AOP, BOQ, COR, DOS equal to $\frac{1}{2} C$, $\frac{1}{2} D$, $\frac{1}{2} A$, $\frac{1}{2} B$, respectively; prove that each side is divided into segments proportional to its adjacent sides. Thus—

$$AP : PB :: d : b, \text{ \&c.}$$

15. Prove that the points P, Q, R, S are concyclic.

16. ,, $OP = \sqrt{abcd}/(a + c)$.

17. If the lengths of the sides remain constant while the quadrilateral alters its shape; prove, when it becomes inscriptible, that P, Q, R, S will be the points of contact of the incircle.

18. If OP' be the isogonal conjugate of OP with respect to the angle AOB; prove that

$$AP' = AO^2/d, \quad BP' = CO^2/b. \qquad \text{(Malet.)}$$

19. If R be the circumradius of a cyclic quadrilateral, prove that

$$4R = \sqrt{\frac{(ac + bd)(ab + cd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}.$$

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* The theorems 12–17 have been communicated to the author by Mr. W. S. M'Kay, F.T.C.D.

CHAPTER VII.

CONTINUATION OF THE THEORY OF CIRCULAR FUNCTIONS.

SECTION I.—DE MOIVRE'S THEOREM.

155. *Every quantity of the form* $a + bi$, *where* $i = \sqrt{-1}$, *denotes the square root of negative unity can be put under the form* $\rho\,(\cos\alpha + i\sin\alpha)$.

In fact, it suffices to have $\rho\cos\alpha = a$, $\rho\sin\alpha = b$;

whence $\rho = \sqrt{a^2 + b^2}$, $\cos\alpha = a/\rho$, $\sin\alpha = b/\rho$.

ρ is called the *modulus:* we always take it positive; $a + bi$ is called a *complex magnitude.* If $\rho = 1$, $a + bi$ is a *unimodular complex.* The equations

$$\cos\alpha = a/\rho, \quad \sin\alpha = b/\rho,$$

determine an angle ϕ uniquely between 0 and 2π;* then $\alpha = \phi + 2n\pi$ is called the *argument*; it is determined, except a multiple of 2π.

Cor.—In order that two complex magnitudes may be equal, it is necessary that their moduli be equal, and that their arguments differ only by a multiple of 2π.

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* These two equations are necessary in order to determine ϕ. The equation $\tan\phi = \frac{a}{b}$, which is inferred from them, is insufficient.

156. *The product of several unimodular complexes is an unimodular complex whose argument is equal to the sum of their arguments.*

Dem.—Multiplying $\cos\alpha + i\sin\alpha$ by $\cos\beta + i\sin\beta$, we get

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Again, multiylping both sides by $\cos\gamma + i\sin\gamma$, we get

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$
$$= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma), \qquad (474)$$

and so on for the product of any number of factors.

Cor. 1.—*The quotient of two unimodular complexes is an unimodular complex whose argument is equal to the difference of their arguments.*

From the theorem of this section we can get an expression for the tangent of the sum of any number of angles. For writing it in the form

$$\cos(\alpha+\beta\ldots\lambda) + i\sin(\alpha+\beta\ldots\lambda)$$

$$= (\cos\alpha\cos\beta\ldots\cos\lambda)(1+i\tan\alpha)(1+i\tan\beta)\ldots(1+i\tan\lambda).$$

And, multiplying the factors

$$(1+i\tan\alpha), \quad (1+i\tan\beta), \text{ \&c.,}$$

and comparing real and imaginary parts, we get

$$\cos(\alpha+\beta\ldots\lambda) = (\cos\alpha.\cos\beta\ldots\cos\lambda)(1 - s_2 + s_4 - s_6, \text{ \&c.}),$$

$$\sin(\alpha+\beta\ldots\lambda) = (\cos\alpha.\cos\beta\ldots\cos\lambda)(s_1 - s_3 + s_5, \text{ \&c.}),$$

where s_1, s_2, s_3, &c., denote the sumof $\tan\alpha$, $\tan\beta\ldots\tan\lambda$, the sum of their combinations 2 by 2, 3 by 3, &c. Hence, by division, we get

$$\tan(\alpha+\beta\ldots\lambda) = \frac{s_1 - s_3 + s_5\ldots}{1 - s_2 + s_4\ldots}. \quad \text{(Compare § 48.)}$$

157. *If n be any number, positive or negative, integral or fractional,*

$$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha.$$

1°. *Let n be positive and integral.*

We have (§ 156),

$$(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \ldots (\cos \lambda + i \sin \lambda)$$
$$= \cos (\alpha + \beta \ldots \lambda) + i \sin (\alpha + \beta \ldots \lambda).$$

Suppose now that the arguments α, β, ... λ are all equal, and there are n factors, we get

$$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha.$$

2°. *Let the exponent be negative and integral.*

We have, by the theory of indices,

$$(\cos \alpha + i \sin \alpha)^{-n} = \frac{1}{(\cos \alpha + i \sin \alpha)^n} = \frac{1}{\cos n\alpha + i \sin n\alpha} \text{ by } (1°)$$
$$= \frac{\cos^2 n\alpha + \sin^2 n\alpha}{\cos n\alpha + i \sin n\alpha} = \cos n\alpha - i \sin n\alpha.$$

Hence $$(\cos \alpha + i \sin \alpha)^{-n} = \cos (-n\alpha) + i \sin (-n\alpha).$$

3°. *Let the exponent be fractional, such as $\frac{p}{q}$.*

Assume $$(\cos \alpha + i \sin \alpha)^{\frac{p}{q}} = \cos x + i \sin x;$$
$$\therefore \quad (\cos \alpha + i \sin \alpha)^p = (\cos x + i \sin x)^q,$$
$$\cos p\alpha + i \sin p\alpha = \cos qx + i \sin qx;$$
$$\therefore \quad qx = p\alpha + 2n\pi, \qquad x = \frac{p\alpha + 2n\pi}{q};$$
$$\therefore \quad (\cos \alpha + i \sin \alpha)^{\frac{p}{q}} = \cos \frac{(p\alpha + 2n\pi)}{q} + i \sin \frac{(p\alpha + 2n\pi)}{q}.$$

If we take $n = 0$, we have

$$(\cos a + i \sin a)^{\frac{p}{q}} = \cos \frac{pa}{q} + i \sin \frac{pa}{q}.$$

We thus have one of the values of

$$(\cos a + i \sin a)^{\frac{p}{q}};$$

but it has q values, as we shall see further on.

Hence, in general,

$$(\cos a + i \sin a)^n = \cos na + i \sin na. \tag{475}$$

In like manner,

$$(\cos a - i \sin a)^n = \cos na - i \sin na. \tag{476}$$

The result just proved (475), (476) is called De Moivre's Theorem, and is a fundamental one in Analytical Mathematics.

Cor.—If n be any positive integer—

(1) $$\cos na = \cos^n a - \frac{n \, . \, n-1}{\lfloor 2} \cos^{n-2} a \sin^2 a + \frac{n \, . \, n-1 \, . \, n-2 \, . \, n \, . \, 3}{\lfloor 4} \cos^{n-4} a \sin^4 a - \&c. \tag{477}$$

(2) $$\sin na = n \cos^{n-1} a \sin a - \frac{n \, . \, n-1 \, . \, n-2}{\lfloor 3} \cos^{n-3} a \sin^3 a + \&c. \tag{478}$$

Expand the left-hand member of (475), and equate real and imaginary parts.

Or thus :—Multiply (477) by $\cos a$, and (478) by $\sin a$, and subtract. Again, multiply (477) by $\sin a$, and (478) by $\cos a$, and add. We thus get corresponding identities for

$$\cos (n + 1) a, \quad \sin (n + 1) a.$$

And, since the equations are true for $n = 1$, they are true for $n = 2, 3,$ &c. Hence they are universally true.

158. *Development of sin x, cos x.*

$$(\cos x + i \sin x) = \left(\cos \frac{x}{n} + i \sin \frac{x}{n}\right)^n;$$

whence, developing the second side, and equating real and imaginary parts, we get

$$\cos x/\cos^n x/n = 1 - \frac{n \, . \, n-1}{\lfloor 2} \tan^2 \frac{x}{n} + \frac{n \, . \, n-1 \, . \, n-2 \, . \, n-3}{\lfloor 4} \tan^4 \frac{x}{n} - \&c.,$$

$$\sin x/\cos^n x/n = n \tan \frac{x}{n} - \frac{n \, . \, n-1 \, . \, n-2}{\lfloor 3} \tan^3 \frac{x}{n} + \&c.$$

Denoting these results by

$$\cos x/\cos^n x/n = 1 - u_2 + u_4 - u_6, \&c., \ldots,$$

$$\sin x/\cos^n x/n = u_1 - u_3 + u_5 - \&c., \ldots,$$

the ratio of a term to the preceding one is of the form

$$\frac{-(n - \overline{p-1})(n-p)}{p \, . \, p+1} . \tan^2 \frac{x}{n}$$

$$= \frac{\left(1 - \frac{p-1}{n}\right)\left(1 - \frac{p}{n}\right)}{p \, . \, p+1} x^2 \left(\frac{\tan x/n}{x/n}\right)^2.$$

If p be sufficiently great, this ratio will be less than 1, for

$$\left(\left(1 - \frac{p-1}{n}\right)\left(1 - \frac{p}{n}\right) < 1\right),$$

and $p\,(p+1)$ can be made greater than

$$x^2 \left(\frac{\tan x/n}{x/n}\right)^2.$$

Hence we infer

$$\cos x/\cos^n x/n > 1 - u_2 + u_4 - u_6 \ldots + u_{4q} - u_{4q+2},$$

and $$< 1 - u_2 + u_4 - u_6 \ldots + u_{4q};$$

but (Exercises xv., 10), limit of $\cos^n \frac{x}{n}$ is 1.* And we have

$$u_{2p} = \pm \frac{n(n-1)\ldots n-2p+1}{\underline{|2p}} \tan^{2p}\frac{x}{n}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{2p-1}{n}\right)}{\underline{|2p}} x^{2p}\left(\frac{\tan x/n}{x/n}\right)^{2p}.$$

Now, if n increase indefinitely, the limit of

$$u_{2p} \text{ is } \frac{x^{2p}}{\underline{|2p}}.$$

Hence, $$\cos x > 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} \ldots - \frac{x^{4q+2}}{\underline{|4q+2}}$$

$$< 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} \ldots + \frac{x^{4q}}{\underline{|4q}};$$

---

* *Or thus :—*

$$1 - \cos\frac{x}{n} < \frac{x^2}{2n^2}.$$

Hence $$\cos\frac{x}{n} - \cos^2\frac{x}{n} < \frac{x^2}{2n^2}, \quad \cos^2\frac{x}{n} - \cos^3\frac{x}{n} < \frac{x^2}{2n^2}, \text{ \&c.}$$

Hence, by addition,

$$1 - \cos^n\frac{x}{n} < n \cdot \frac{x^2}{2n^2}, \text{ that is } < \frac{x^2}{2n};$$

therefore $$\cos^n\frac{x}{n} < 1 \text{ and } > 1 - \frac{x^2}{2n}.$$

Hence the limit is 1.

and, making q increase without limit, we get

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \text{\&c., to infinity.} \qquad (479)$$

Similarly,

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \text{\&c., to infinity.} \qquad (480)$$

These expressions are due to NEWTON. See WILLIAMSON's *Differential Calculus*, p. 66.

159. The series

$$1 + \frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{\&c.},$$

is convergent for all real values of x, and represents e^x (§ 81). It is also convergent for imaginary values of x of the form

$$a + ib = \rho(\cos\theta + i\sin\theta);$$

for, by substitution, the series takes the form

$$1 + \rho(\cos\theta + i\sin\theta) + \frac{\rho^2}{\lfloor 2}(\cos 2\theta + i\sin 2\theta) + \frac{\rho^3}{\lfloor 3}(\cos 3\theta + i\sin 3\theta) + \text{\&c.}$$

But this is the sum of the two series

$$1 + \rho\cos\theta + \frac{\rho^2}{\lfloor 2}\cos 2\theta + \frac{\rho^3}{\lfloor 3}\cos 3\theta, \text{ \&c.},$$

$$1 + \rho\sin\theta + \frac{\rho^2}{\lfloor 2}\sin 2\theta + \frac{\rho^3}{\lfloor 3}\sin 3\theta, \text{ \&c.},$$

which are each convergent, because their terms are less in absolute value than those of the convergent series

$$1 + \rho + \frac{\rho^2}{\lfloor 2} + \frac{\rho^3}{\lfloor 3} + \text{\&c.}$$

DEF.—The series

$$1 + \frac{x + yi}{1} + \frac{(x + yi)^2}{\underline{|2}} + \frac{(x + yi)^3}{\underline{|3}} + \text{\&c.}$$

is what is understood by $e^{(x+yi)}$.

160. Euler's Theorem.

$$\cos\theta + i\sin\theta = e^{i\theta}. \tag{481}$$

For, substituting for $\cos\theta$, $\sin\theta$, their expansions (479), (480), we get what by definition is denoted by the second side.

161. Theorem.

For all values of x and y, real, imaginary, or complex—

$$e^x . e^y = e^{x+y}. \tag{482}$$

DEM.—

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \text{\&c.,} \quad (\S\S\ 81 \text{ and } 159)$$

$$e^y = 1 + \frac{y}{1} + \frac{y^2}{\underline{|2}} + \frac{y^3}{\underline{|3}} + \text{\&c.}$$

Hence, multiplying,

$$\begin{array}{rl|l|l}
e^x . e^y = 1 & + \frac{x}{1} & + \frac{x^2}{\underline{|2}} & + \frac{x^3}{\underline{|3}} + \text{\&c.} \\
& + \frac{y}{1} & + \frac{x}{1} . \frac{y}{1} & + \frac{x^2 y}{\underline{|2}} + \text{\&c.} \\
& & + \frac{y^2}{\underline{|2}} & + \frac{xy^2}{\underline{|2}} + \text{\&c.} \\
& & & + \frac{y^3}{\underline{|3}} + \text{\&c.}
\end{array}$$

$$= 1 + \frac{x + y}{1} + \frac{(x + y)^2}{\underline{|2}} + \frac{(x + y)^3}{\underline{|3}} + \text{\&c.} = e^{x+y}.$$

162. Def.—*If* e^{x+iy} *be equal to* $\alpha + i\beta$, $x + iy$ *is called the Napierian logarithm of* $\alpha + \beta i$.

From the equation $e^{x+iy} = \alpha + i\beta$ we have (§ 161)

$$e^x \cdot e^{iy} = \alpha + i\beta;$$

or, putting $\alpha + i\beta$ (§ 155) $= \rho(\cos\theta + i\sin\theta)$,

$$e^x(\cos y + i\sin y) = \rho(\cos\theta + i\sin\theta).$$

Hence $e^x = \rho$, and $x = \log\rho$ (real logarithm), $y = \theta + 2n\pi$;

therefore $$x + iy = \log\rho + (\theta + 2n\pi)i,$$

or $$\log(\alpha + i\beta) = \log\rho + (\theta + 2n\pi)i. \qquad (483)$$

Hence an imaginary quantity has an infinite number of logarithms.

163. The Binomial Theorem.

If n *be any quantity, whole or fractional, positive or negative, and* x *any quantity, real or complex, whose modulus is* < 1, *then*

$$1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n \cdot n-1 \cdot n-2}{\lfloor 3}x^3 + \&c. = (1+x)^n. \qquad (484)$$

Dem.—If x be real, the series on the right is convergent; for the ratio of the $(p+1)^{th}$ term to the p^{th} term is

$$\left(\frac{n+1}{p} - 1\right)x,$$

which, when p is large, differs very little from x; then it is less than a fixed quantity $a < 1$. Hence (§ 77), the series is convergent. If x be complex, and $= \rho(\cos\alpha + i\sin\alpha)$, the series is composed of two series, viz.—

$$1 + \frac{n}{1}\rho\cos\alpha + \frac{n(n-1)}{\lfloor 2}\rho^2\cos 2\alpha + \frac{n \cdot n-1 \cdot n-2}{\lfloor 3}\rho^3\cos 3\alpha + \&c.$$

$$+ i\left(\frac{n}{1}\rho\sin\alpha + \frac{n \cdot n-1}{\lfloor 2}\rho^2\sin 2\alpha + \frac{n \cdot n-1 \cdot n-2}{\lfloor 3}\rho^3\sin 3\alpha + \&c.\right),$$

which are convergent, because their terms are respectively less than those of the convergent series—

$$\frac{n}{1}\rho + \frac{n \,.\, n-1}{\lfloor 2}\rho^2 + \frac{n \,.\, n-1 \,.\, n-2}{\lfloor 3}\rho^3.$$

Now, put

$$\phi(n) = 1 + nx + \frac{n \,.\, n-1}{\lfloor 2}x^2 + \frac{n \,.\, n-1 \,.\, n-2}{\lfloor 3}x^3 + \&c. \quad (1);$$

and, changing n into n_1, we have

$$\phi(n_1) = 1 + n_1 x + \frac{n_1(n_1-1)}{\lfloor 2}x^2 + \frac{n_1 \,.\, n_1-1 \,.\, n_1-2}{\lfloor 3}x^3 + \&c.$$

Hence, by multiplication,

$$\begin{array}{rl|l|l|l}
\phi(n)\phi(n_1) = 1 + nx & + \frac{n \,.\, n-1}{\lfloor 2}x^2 & + \frac{n.n-1 \,.\, n-2}{\lfloor 3}x^3 & +\&c. \\
n_1 x & + nn_1x^2 & + \frac{n_1 n(n-1)}{\lfloor 2}x^3 & +\&c. \\
 & + \frac{n_1(n_1-1)}{\lfloor 2}x^2 & + \frac{n.n_1(n_1-1)}{\lfloor 2}x^3 & +\&c. \\
 & & + \frac{n_1.(n_1-1)(n_1-2)}{\lfloor 3}x^3 & +\&c.
\end{array}$$

$$= 1 + (n+n_1)x + \frac{(n+n_1)(n+n_1-1)}{\lfloor 2}x^2$$
$$+ \frac{(n+n_1)(n+n_1-1)(n+n_1-2)}{\lfloor 3}x^3 + \&c.$$

Hence, $\qquad \phi(n)\phi(n_1) = \phi(n+n_1);$

and, multiplying this by $\phi(n_2)$, we get

$$\phi(n)\phi(n_1)\phi(n_2) = \phi(n+n_1+n_2),$$

and, in general,

$$\phi(n)\phi(n_1)\phi(n_2)\,.\,.\,.\,\phi(n_{p-1}) = \phi(n+n_1+n_2\,.\,.\,.\,n_{p-1}).$$

In this equation, the quantities n, n_1, n_2, &c., may be whole or

fractional, positive or negative. Supposing them all equal, we get

$$[\phi(n)]^p = \phi(np). \quad (1)$$

Hence, changing n into $\pm n/p$, we have

$$[\phi(\pm n/p)]^p = \phi(\pm n) = \phi(\pm 1)^n.$$

Now $\qquad \phi(+1) = 1 + x,$ from (1),

$$\phi(-1) = 1 - x + x^2 - x^3 + \&c. = \frac{1}{1+x} = (1+x)^{-1};$$

therefore $\qquad \phi(\pm n/p)^p = (1+x)^{\pm n/p}.$

Hence $\phi(\pm n/p)$ is one of the values of $(1+x)^{\pm n/p}$. And the proposition is proved.

The foregoing proof, taken from SERRET's *Trigonometry*, is, with slight modification, the same as that given by EULER, Tome xix., *Novi Comment. Acad. Petrop.*, and which Lacroix says—"Tient le premier rang par sa finesse et sa brièvete." For an exhaustive discussion, see ABEL's Works, Tome i., p. 219.

Cor.—From the generalised form of the binomial theorem given in this section we can prove, as in § 84, that the expansion of $\log_e(1+x)$, where x denotes any quantity, real or complex, whose modulus is < 1, is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}, \ \&c.$$

164. Gregory's Series.

From Euler's theorem, § 160, we have

$$e^{i\theta} = \cos\theta(1 + i\tan\theta).$$

Hence, $\qquad i\theta = \log\cos\theta + \log(1 + i\tan\theta)$

$$= \log\cos\theta + \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \&c.$$

$$+ i\left(\tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \&c.\right);$$

and, comparing real and imaginary parts, we get

$$\log \cos\theta + \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \&c. = 0, \tag{485}$$

and $$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \&c. \tag{486}$$

Since the modulus of x in the expansion of $\log_e(1 \pm x)$ must be < 1, it follows that $\tan\theta$ in the expansion of $\log_e(1 + i\tan\theta)$ must be between $+ 1$ and $- 1$; *and hence, in equations* (485), (486), *θ must lie between* $+\frac{\pi}{4}$ *and* $-\frac{\pi}{4}$.

In equation (486) put $\tan\theta = u$, and we get

$$\tan^{-1}u = u - \frac{u^3}{3} + \frac{u^5}{5} - \&c.,$$

which is Gregory's series. (487)

The series (485) may be proved as follows:—

$$\cos^2\theta\,(1 + \tan^2\theta) = 1;$$

and, taking logarithms, we get the series required.

165. In Euler's theorem,

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

Change θ into $- \theta$, and we get

$$\cos\theta - i\sin\theta = e^{-i\theta}.$$

Hence, taking the sum and difference, we have

$$2\cos\theta = e^{i\theta} + e^{-i\theta}. \tag{488}$$

$$2i\sin\theta = e^{i\theta} - e^{-i\theta}. \tag{489}$$

166. In Euler's theorem (§ 160), put $\theta = 2n\pi$, and then $= (2n + 1)\pi$, and we get

$$1 = e^{i(2n\pi)}, \quad \text{and } - 1 = e^{i(2n+1)\varpi}.$$

Hence $\log_e(1) = i(2n\pi)$, and $\log_e(-1) = i(2n + 1)\pi$. (490)

Hence, positive unity and negative unity have each an infinite number of imaginary logarithms.

167. The most important application of Gregory's series is to find the value of π. Thus, if in equation (487) we put

$$\tan^{-1} u = \frac{\pi}{4}, \quad \text{then} \quad u = 1,$$

and we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}, \text{ \&c.} \tag{491}$$

This series converges too slowly to be used in calculation; but if in the equation

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4} \quad (\S\ 66, \text{ Ex. } 3.)$$

we substitute, from Gregory's series, for

$$\tan^{-1}\frac{1}{2}, \quad \tan^{-1}\frac{1}{3},$$

their values, we get a result called Euler's Series, which is more convergent, viz.—

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}\right) - \frac{1}{7}\left(\frac{1}{2^7}\right) + \text{ \&c.}$$

$$+ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{3^5}\right) - \frac{1}{7}\left(\frac{1}{3^7}\right) + \text{ \&c.} \tag{492}$$

A still more convergent series, called Machin's, is obtained from the equation

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}. \quad (\S\ 66, \text{ Ex. } 5.)$$

Since $$\tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99},$$

Machin's series may be transformed into one depending on powers of $\frac{1}{70}$ and $\frac{1}{99}$, and therefore more easily calculated. Two English computers, Dr. Rutherford and Mr. William Shanks, calculated π to 500 and 707 places of decimals, respectively.—*Proceedings of the Royal Society*, vols. xxi., xxii.

168. The value of π may be written in the form of a continual fraction. Thus, assume

$$\frac{\pi}{4} = \frac{\alpha}{a+}\ \frac{\beta}{b+}\ \frac{\gamma}{c+}\ \frac{\delta}{d+}, \text{ \&c.},$$

and equate the differences of the successive convergents to

$$\frac{1}{3},\ \frac{1}{5},\ \frac{1}{7}, \text{ \&c.},$$

and we easily get

$$\frac{\pi}{4} = \frac{1}{1+}\ \frac{1}{2+}\ \frac{9}{2+}\ \frac{25}{2+}\ \frac{49}{2+}, \text{ \&c.} \tag{493}$$

Exercises.—XXXV.

1. If $2\cos\theta = x + \frac{1}{x}$, prove $x - \frac{1}{x} = 2i\sin\theta$. (494)

2. ,, ,, ,, $2\cos m\theta = x^m + \frac{1}{x^m}$, and $2i\sin m\theta = x^m - \frac{1}{x^m}$. (495)

3. If $2\cos\theta = x + \frac{1}{x}$, and $2\cos\phi = y + \frac{1}{y}$, prove

$$2\cos(m\theta \pm n\phi) = x^m y^n \mp \frac{1}{x^m y^n}. \tag{496}$$

4. Decompose $1 + c\cos\theta$ into the product of two imaginary factors, when c is < 1.

$$\left[\text{Put } c = \frac{2n}{1+n^2}\right].$$

5. Prove $\sin(x+h) = \sin x + h\cos x - \frac{h^2}{\lfloor 2}\sin x + \frac{h^3}{\lfloor 3}\cos x$, &c. (497)

6. Prove the following expression for finding approximately the length of an arc of a circle, viz., $\frac{8c - c'}{3}$; where c, c' denote respectively the chord of half the arc, and of the whole arc. (Huyghens.)

7. Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$; prove $\theta = 2^\circ$ nearly.

8. Given $\sin\theta = n\sin(\theta + \alpha)$; prove

$$\theta = n\sin\alpha + \frac{n^2}{2}\sin 2\alpha + \frac{n^3}{3}\sin 3\alpha + \&c. \tag{498}$$

Exponential values being used, we get

$$e^{2\theta} = \frac{1 - ne^{-i\alpha}}{1 - ne^{i\alpha}};$$

and, taking logarithms, we get the required result.

9. Prove

$$\log_e\sqrt{(1 + 2\rho\cos\theta + \rho^2)} = \rho\cos\theta - \frac{\rho^2\cos 2\theta}{2} + \frac{\rho^3\cos 3\theta}{3} - \&c., \tag{499}$$

and
$$\tan^{-1}\frac{\rho\sin\theta}{1 + \rho\cos\theta} = \rho\sin\theta - \frac{\rho^2\sin 2\theta}{2} + \frac{\rho^3\sin 3\theta}{3} - \&c. \tag{500}$$

Put
$$1 + \rho\cos\theta = r\cos\phi \text{ and } \rho\sin\theta = r\sin\phi,$$

and we get

$$r = \sqrt{1 + 2\rho\cos\theta + \rho^2}, \quad \phi = \tan^{-1}\frac{\rho\sin\theta}{1 + \rho\cos\theta}, \text{ and } re^{i\phi} = 1 + \rho e^{i\theta}.$$

Hence, taking logarithms, and substituting for $e^{i\theta}$, $e^{2i\theta}$, &c., their values from Euler's Theorem, we get the required results.

10. Prove

$$\log_e\sqrt{1 - 2\rho\cos\theta + \rho^2} = -\left(\rho\cos\theta + \frac{\rho^2\cos 2\theta}{2} + \frac{\rho^3\cos 3\theta}{3} + \&c.\right), \tag{501}$$

and
$$\tan^{-1}\frac{\rho\sin\theta}{1 - \rho\cos\theta} \doteq \rho\sin\theta + \frac{\rho^2\sin 2\theta}{2} + \frac{\rho^3\sin 3\theta}{3}, \&c. \tag{502}$$

11. If n be any positive integer; prove that

$$2^{n-1}\cos^n\alpha = \cos n\alpha + n\cos(n-2)\alpha + \frac{n \cdot n-1}{\lfloor 2}\cos(n-4)\alpha + \&c., \tag{503}$$

the number of terms being

$$\frac{n}{2} + 1, \quad \text{or} \quad \frac{n+1}{2},$$

according as n is even or odd; and when n is even, only half the last term to be taken.

DEM.—Put $$2\cos\alpha = x + \frac{1}{x};$$

$$\therefore\quad 2^n \cos^n\alpha = x^n + \frac{1}{x^n} + n\left(x^{n-2} + \frac{1}{x_{n-2}}\right) + \frac{n\,.\,n-1}{\underline{|2}}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.,$$

and substituting for

$$x^n + \frac{1}{x^n},\quad x^{n-2} + \frac{1}{x^{n-2}},\ \&c.,$$

their values.

Or thus:—Multiplying both sides by $2\cos\alpha$, and making use of the identity

$$2\cos n\alpha\cos\alpha = \cos(n+1)\alpha + \cos(n-1)\alpha,$$

we see that if the proposition be true for any assumed value, it will be true for the next highest; but it is evidently true for $n = 2$. Hence it is true for $n = 3$, &c.

12. Prove $$2\cos n\alpha = (2\cos\alpha)^n - n(2\cos\alpha)^{n-2} + \frac{n\,.\,n-3}{\underline{|2}}(2\cos\alpha)^{n-4},\ \&c., \tag{504}$$

the series on the right to continue so long as the powers of $2\cos\alpha$ are positive.

DEM.—Write down the right-hand side of (299) in the form

$$\frac{\cos\alpha - x}{1+x^2}\left(1 - \frac{2x\cos\alpha}{1+x^2}\right)^{-1},$$

expand, and compare coefficients of x^{n-1} on both sides.

13. Prove, when n is odd, that

$$(-1)^{\frac{n-1}{2}}\, 2^{n-1}\sin^n\alpha = \sin n\alpha - n\sin(n-2)\alpha + \frac{n\,.\,n-1}{\underline{|2}}\sin(n-4)\alpha,\ \&c. \tag{505}$$

This may be proved like Ex. 11, by De Moivre's theorem, or independently. *Thus*:—Multiply both sides by $(-1)^2\sin^2\alpha$, and use the identity

$$2^2\sin(2m-1)\alpha\sin^2\alpha = 2\sin(2m-1)\alpha - \sin(2m+1)\alpha - \sin(2m-3)\alpha,$$

and if the proposition be true, &c.

14. When n is even, prove

$$(-1)^{\frac{n}{2}}\, 2^{n-1}\sin^n\alpha = \cos n\alpha - n\cos(n-2)\alpha + \frac{n\,.\,n-1}{\underline{|2}}\cos(n-4)\alpha,\ \&c., \tag{506}$$

the last term of which is only numerical, and its half to be taken.

15. Show that, commencing with the last term in equation (504), we get, writing the series backwards,

$$\cos n\alpha = (-1)^{\frac{n}{2}} \left\{1 - \frac{n^2}{\lfloor 2} \cos^2\alpha + \frac{n^2(n^2-2^2)}{\lfloor 4} \cos^4\alpha - \&c.\right\},$$

when n is even, and (507)

$$\frac{\cos n\alpha}{n} = (-1)^{\frac{n-1}{2}} \left\{\cos\alpha - \frac{n^2-1^2}{\lfloor 3} \cos^3\alpha + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 5} \cos^5\alpha - \&c.\right\},$$

when n is odd. (508)

16. Given $\tan l' = n \tan l$, and $m = \frac{1-n}{1+n}$, prove that

$$l' = l - m \sin 2l + \tfrac{1}{2}m^2 \sin 4l - \tfrac{1}{3}m^3 \sin 6l, \ \&c. \tag{509}$$

17. Prove that e^x is equal to the continued fraction—

$$\frac{1}{1-}\ \frac{x}{1+}\ \frac{x}{2-}\ \frac{x}{3+}\ \frac{x}{2-}\ \frac{x}{5+}\ \frac{x}{2-}\ \frac{x}{7+},\ \&c. \tag{510}$$

18. ,, $$\log_e(1+x) = \frac{x}{1+}\ \frac{x}{2+}\ \frac{x}{3+}\ \frac{2x}{2+}\ \frac{2x}{5+}\ \frac{3x}{2+}\ \frac{3x}{7+}. \tag{511}$$

19. ,, $$\tan x = \frac{x}{1-}\ \frac{x^2}{3-}\ \frac{x^2}{5-}\ \frac{x^2}{7-},\ \&c. \tag{512}$$

20. Transform the identity

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} - 1 = 0$$

into

$$\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} \sin 2(\theta-\gamma) + \frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} \sin 2(\theta-\alpha)$$

$$+ \frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)} \sin 2(\theta-\beta) = 0. \tag{513}$$

Assume $x = \cos 2\theta + i \sin 2\theta$, with corresponding values for a, b, c, and then equate the real and the imaginary parts to zero.

21. Transform the identity

$$(a-b)(c-d) + (b-c)(a-d) + (c-a)(b-d) = 0$$

into

$$\sin(\alpha-\beta)\sin(\gamma-\delta) + \sin(\beta-\gamma)\sin(\alpha-\delta) + \sin(\gamma-\alpha)\sin(\beta-\delta) = 0.$$

22. From $\dfrac{a^2}{(a-b)(a-c)(a-d)} + \dfrac{b^2}{(b-a)(b-c)(b-d)} + \&c. = 0,$
prove that

$$\Sigma \frac{\cos 2\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\sin(\alpha-\delta)} = 0, \quad \Sigma \frac{\sin 2\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\sin(\alpha-\delta)} = 0. \tag{514}$$

23. If α, β, $\gamma \ldots \lambda$ be any number of given angles, and θ a variable angle, prove

$$\frac{\sin(\alpha-\beta)}{\sin(\theta-\alpha)\sin(\theta-\beta)} + \frac{\sin(\beta-\gamma)}{\sin(\theta-\beta)\sin(\theta-\gamma)} \cdots \frac{\sin(\lambda-\alpha)}{\sin(\theta-\lambda)\sin(\theta-\alpha)} = 0. \tag{515}$$

24. When n is even, prove that

$$\cos n\alpha = 1 - \frac{n^2}{\lfloor 2}\sin^2\alpha + \frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\alpha - \&c. \tag{516}$$

[In (507) change α into $\frac{\pi}{2} - \alpha$.]

25. When n is odd, prove

$$\sin n\alpha = n\left\{\sin\alpha - \frac{n^2-1^2}{\lfloor 3}\sin^3\alpha + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 5}\sin^5\alpha - \&c.\right\} \tag{517}$$

[In (508) change α into $\frac{\pi}{2} - \alpha$.]

26. When n is even, prove

$$\sin n\alpha = n\cos\alpha\left\{\sin\alpha - \frac{n^2-2^2}{\lfloor 3}\sin^3\alpha + \frac{(n^2-2^2)(n^2-4^2)}{\lfloor 5}\sin^5\alpha - \&c.\right\} \tag{518}$$

[Differentiate (516).]

27. When n is odd, prove

$$\cos n\alpha = \cos\alpha\left\{1 - \frac{n^2-1^2}{\lfloor 2}\sin^2\alpha + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4}\sin^4\alpha - \&c.\right\} \tag{519}$$

[Differentiate (517).]

28. If $\dfrac{a}{b} = \tan\theta$, and $a^2 + b^2 = \rho^2$, prove that

$$e^{ax}\cos bx = 1 + (\rho\sin\theta)x + (\rho^2\sin 2\theta)\frac{x^2}{\lfloor 2} + (\rho^3\sin 3\theta)\frac{x^3}{\lfloor 3} + \&c. \tag{520}$$

[Substitute its exponential value for $\cos bx$, and expand.]

P 2

29. In any plane triangle,

$$B = \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C, \text{ \&c.} \tag{521}$$

[Make use of equation (498), and $\sin B = \frac{b}{a}\sin(B + C)$.]

30. In any plane triangle,

$$\log a - \log c = \frac{b}{a}\cos C + \frac{b^2}{2a^2}\cos 2C + \frac{b^3}{3a^3}\cos 3C + \text{\&c.} \tag{522}$$

[Substitute its exponential value for $\cos C$ in $a^2 + b^2 - 2ab\cos C = c^2$.]

31–42. Bernoulli's Numbers.

31. Given $\quad 1 - \frac{\theta}{2}\cot\frac{\theta}{2} = B_1\frac{\theta^2}{\lfloor 2} + B_2\frac{\theta^4}{\lfloor 4} + B_3\frac{\theta^6}{\lfloor 6}$, &c., $\quad (\alpha)$

prove that

$$\frac{1}{\lfloor 2n+1} - \frac{1}{\lfloor 2\,\lfloor 2n} + \frac{B_1}{\lfloor 2\,\lfloor 2n-1} - \frac{B_2}{\lfloor 4\,\lfloor 2n-3} + \text{\&c.} \pm \frac{B_n}{\lfloor 2n} = 0. \tag{523}$$

Multiply equation (α) by $\sin\theta$, and we get

$$\sin\theta - \frac{\theta}{2}(1 + \cos\theta) = \left(B_1\frac{\theta^2}{\lfloor 2} + B_2\frac{\theta^4}{\lfloor 4} + B_3\frac{\theta^6}{\lfloor 6}, \text{ \&c.}\right)\sin\theta.$$

In this identity, substitute for $\sin\theta$ and $\cos\theta$ their Newtonian expansions, and equate the coefficients of θ^{2n+1}.

32. In the same case, prove

$$\frac{1}{\lfloor 2n+2} - \frac{1}{\lfloor 2\,\lfloor 2n+1} + \frac{B_1}{\lfloor 2\,\lfloor 2n} - \frac{B_2}{\lfloor 4\,\lfloor 2n-2} \cdots \pm \frac{B_n}{\lfloor 2n\,\lfloor 2} = 0. \tag{524}$$

[Make use of the identity

$$\left(1 - \frac{\theta}{2}\cot\frac{\theta}{2}\right)(1 - \cos\theta) = 1 - \cos\theta - \frac{\theta}{2}\sin\theta,$$

and equate coefficients of θ^{2n+2}.]

The quantities B_1, B_2, B_3, &c., are called Bernoulli's numbers. They occur in several investigations. The proofs in (523), (524) are from Serret's *Trigonometry*, p. 259.

33. Prove that $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$. (525)

34. ,, $\quad \dfrac{\theta}{e^\theta - 1} = 1 - \dfrac{\theta}{2} + \dfrac{B_1\theta^2}{\lfloor 2} - \dfrac{B_2\theta^4}{\lfloor 4} + \dfrac{B_3\theta^6}{\lfloor 6} - \text{\&c}$ (526)

35. ,, $$\frac{\theta}{e^\theta+1}=\frac{\theta}{2}-\frac{(2^2-1)}{\lfloor\underline{2}}B_1\theta^2+\frac{(2^4-1)}{\lfloor\underline{4}}B_2\theta^4-\&c. \qquad (527)$$

36. ,, $$\tan\theta=\frac{2^2(2^2-1)}{\lfloor\underline{2}}B_1\theta+\frac{2^4(2^4-1)}{\lfloor\underline{4}}B_2\theta^3+\frac{2^6(2^6-1)}{\lfloor\underline{6}}B_3\theta^5, \&c. \qquad (528)$$

In the identity,

$$1-\frac{\theta}{2}\cot\frac{\theta}{2}=B_1\frac{\theta^2}{\lfloor\underline{2}}+B_2\frac{\theta^4}{\lfloor\underline{4}}+B_3\frac{\theta^6}{\lfloor\underline{6}}+\&c. \quad \text{(Ex. 31.)}$$

Change θ into 2θ, and we get

$$\cot\theta=\frac{1}{\theta}-2^2B_1\frac{\theta}{\lfloor\underline{2}}-2^4B_2\frac{\theta^4}{\lfloor\underline{4}}-2^6B_3\frac{\theta^6}{\lfloor\underline{6}}, \&c. \qquad (529)$$

Again, change θ into 2θ, and multiply by 2, and we get

$$2\cot 2\theta=\frac{1}{\theta}-2^4B_1\frac{\theta}{\lfloor\underline{2}}-2^8B_2\frac{\theta^3}{\lfloor\underline{4}}-2^{12}B_3\frac{\theta^5}{\lfloor\underline{5}}, \&c.;$$

but $$\tan\theta=\cot\theta-2\cot 2\theta.$$

Hence the proposition is proved.

37. Prove $$\frac{e^\theta-1}{e^\theta+1}=(2^2-1)B_1\theta+\frac{2(2^4-1)}{\lfloor\underline{4}}B_2\theta^3+\frac{2(2^6-1)}{\lfloor\underline{6}}B_3\theta^5+\&c. \qquad (530)$$

[Make use of (528).]

38. ,, $$\frac{\theta\sin 3\theta}{\sin\theta\,.\,\sin 2\theta}=\frac{3}{2}-\frac{2^2+2^3}{\lfloor\underline{2}}B_1\theta^2-\frac{2^4+2^7}{\lfloor\underline{4}}B_2\theta^4-\frac{2^6+2^{11}}{\lfloor\underline{6}}B_3\theta^6. \qquad (531)$$

$$\left[\frac{\sin 3\theta}{\sin\theta\sin 2\theta}=\cot\theta+\cot 2\theta.\right]$$

39. ,, $$\frac{1}{\lfloor\underline{2n}}-\frac{1}{\lfloor\underline{2n+1}}-\frac{2^2B_1}{\lfloor\underline{2}\,\lfloor\underline{2n-1}}+\frac{2^4B_2}{\lfloor\underline{4}\,\lfloor\underline{2n-3}}-\&c.$$

$$+\frac{(-1)^n2^{2n}B_n}{\lfloor\underline{2n}}=0. \qquad (532)$$

[$\cos\theta=\cot\theta\sin\theta$.]

40. Prove $$\frac{1}{\lfloor 2n-1} - \frac{2^2(2^2-1)}{\lfloor 2 \lfloor 2n-2} B_1 + \frac{2^4(2^4-1)}{\lfloor 4 \lfloor 2n-4} B_2 - \&c.$$

$$+ \frac{(-1)^n 2^{2n}(2^{2n}-1)}{\lfloor 2n} B_n = 0. \quad (533)$$

$$[\sin\theta = \tan\theta \cos\theta.]$$

41. ,, $$\operatorname{cosec}\theta = \frac{1}{\theta} + \frac{2^2-2}{\lfloor 2} B_1 \theta \ldots \frac{2^{2n}-2}{\lfloor 2n} B_n \theta^{2n-1} + \&c. \quad (534)$$

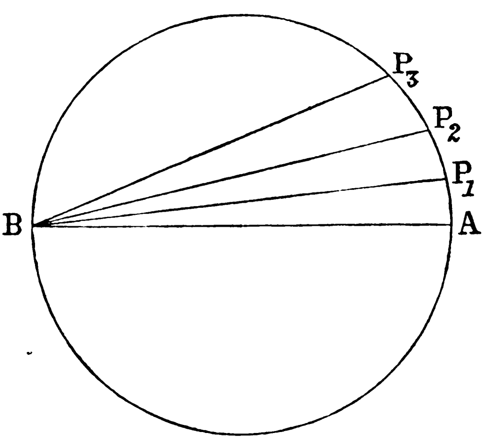
$$[2 \operatorname{cosec}\theta = \tan\tfrac{1}{2}\theta + \cot\tfrac{1}{2}\theta.]$$

42. ,, $$\frac{1}{\lfloor 2n} - \frac{1}{\lfloor 2n-1} + \frac{2^2 B_1}{\lfloor 2 \lfloor 2n-2} - \frac{2^4 B_2}{\lfloor 4 \lfloor 2n-4} + \&c.$$

$$- \frac{(-1)^{n-1} 2^{2n-2} B_{n-1}}{\lfloor 2n-2 \lfloor 2} = \frac{(-1)^n (2^{2n+1}-2)}{\lfloor 2n} B_n. \quad (535)$$

[Use the identity $\cot\theta\cos\theta = \operatorname{cosec}\theta - \sin\theta$, and equations (529), (534).]

43. Vieta's Property of Chords.—*If AB be a diameter of the unit circle, and the arcs AP_1, P_1P_2, P_2P_3, &c., be all equal; then, if the chord BP_1 be denoted by $x + \frac{1}{x}$, BP_2 will be $= x^2 + \frac{1}{x^2}$, $BP_3 = x^3 + \frac{1}{x^3}$,* &c.



Dem.—Denote the angle ABP_1 by θ; then $BP_1 = 2\cos\theta$.

Hence $$x + \frac{1}{x} = 2\cos\theta = e^{i\theta} + e^{-i\theta}; \quad \therefore\ x = e^{i\theta}.$$

Again, $$BP_2 = 2\cos 2\theta = e^{2i\theta} + e^{-2i\theta} = x^2 + \frac{1}{x^2}, \ \&c.$$

Section II.—Binomial Equations.

169. *To solve the binomial equation* $x^n = \pm 1$.

1°. Take the positive sign; and, for greater clearness, let us take a numerical example $x^7 = 1$. Now, putting

$$x = \cos\theta + i\sin\theta,$$

we have, by De Moivre's theorem,

$$x^7 = \cos 7\theta + i\sin 7\theta;$$

$$\therefore \quad \cos 7\theta + i\sin 7\theta = 1;$$

and comparing real and imaginary parts, we get $\cos 7\theta = 1$;

$$\therefore \quad 7\theta = 2\lambda\pi, \text{ and } x = \cos\frac{2\lambda\pi}{7} + i\sin\frac{2\lambda\pi}{7} = e^{\frac{2i\lambda\pi}{7}}.$$

The seven values of x are obtained by giving λ the values 0, 1, 2 . . . 6, respectively. Thus the 7 roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$, where

$$\alpha = e^{\frac{2i\pi}{7}}.$$

Hence the roots are in GP. It is evident that if we give higher values to λ, we get the same roots repeated over again.

If we take the general case $x^n = 1$, the roots are $1, \alpha, \alpha^2 \ldots \alpha^{n-1}$, where

$$\alpha = e^{\frac{2i\pi}{n}}.$$

If n be even, say $2m$, the root α^m will be real; its value will be $e^{i\pi}$; that is, -1. *Hence, when n is even, the equation $x^n = 1$ has two real roots, viz.* $+1$, -1; *but when n is odd, only one root is real.*

2°. Let us take the negative sign. Thus, $x^7 = -1$; then, as before, we get

$$\cos 7\theta + i\sin 7\theta = -1.$$

Hence, evidently, $\qquad 7\theta = (2\lambda + 1)\pi;$

$$\therefore \quad x = \frac{\cos(2\lambda+1)\pi}{7} + \frac{i\sin(2\lambda+1)\pi}{7};$$

that is,
$$x = e^{\frac{i(2\lambda+1)\pi}{7}};$$

and giving λ the values 0, 1, 2 .... 6, we get for x

$$a,\ a^3,\ a^5,\ a^7,\ a^9,\ a^{11},\ a^{13}, \text{ where } a = e^{\frac{i\pi}{7}}.$$

Hence, in this case also, the roots are in GP. It is plain that a^{13}, a^{11}, a^9 may be written a^{-1}, a^{-3}, a^{-5}. In the general case, the roots of $x^n = -1$ are a, a^3, a^5 .... a^{2n-1}, where

$$a = e^{\frac{i\pi}{n}}.$$

When n is odd, there will be one real root, namely, the middle term of the GP, the value of which will be -1. *When n is even, all the roots are imaginary.*

170. *To resolve* $x^n - 1$ *into factors when n is odd.*

Let $n = 2m+1$; then the roots of $x^n - 1 = 0$ are $1, a, a^2 \dots a^{2m}$, and one factor is $x - 1$; and taking the 2nd root and the last, the corresponding factors are $x - a$ and $x - a^{2m}$ or $x - a^{-1}$, and their product is $x^2 - (a + a^{-1})x + 1$; or, since

$$a = e^{\frac{2i\pi}{n}},$$

the product is
$$x^2 - 2\left(\cos\frac{2\pi}{n}\right)x + 1.$$

Similarly, from the roots a^2, a^{2m-1}, we get the quadratic factor

$$x^2 - 2\left(\cos\frac{4\pi}{n}\right)x + 1, \text{ \&c.}$$

Hence, if n be odd,

$$x^n - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{n}.x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{n}.x + 1\right)\dots$$
$$\left(x^2 - 2\cos\frac{(n-1)\pi}{n}.x + 1\right). \qquad (536)$$

Similarly, we may prove the three following decompositions :—

1°. When n is odd,

$$x^n+1=(x+1)(x^2-2\cos\frac{\pi}{n}.x+1)(x^2-2\cos\frac{3\pi}{n}x+1)\ldots$$
$$(x^2-2\cos\frac{n-2}{n}\pi.x+1). \qquad (537)$$

2°. When n is even,

$$x^n-1=(x^2-1)(x^2-2\cos\frac{2\pi}{n}.x+1)(x^2-2\cos\frac{4\pi}{n}.x+1)\ldots$$
$$(x^2-2\cos\frac{n-2}{n}\pi.x+1). \qquad (538)$$

3°. When n is even,

$$x^n+1=(x^2-2\cos\frac{\pi}{n}.x+1)(x^2-2\cos\frac{3\pi}{n}.x+1)\ldots$$
$$(x^2-2\cos\frac{(n-1)\pi}{n}.x+1). \qquad (539)$$

171. *To resolve $x^{2n}-2x^n\cos a+1=0$ into quadratic factors.* The given equation may be written

$$x^{2n}-(e^{ia}+e^{-ia})x^n+1=0, \text{ or } (x^n-e^{ia})(x^n-e^{-ia})=0;$$

that is,

$$\{x^n-(\cos a+i\sin a)\}\{x^n-(\cos a-i\sin a)\}=0,$$

or

$$\{x^n-[\cos(2k\pi+a)+i\sin(2k\pi+a)]\}$$
$$\times\{x^n-[\cos(2k\pi+a)-i\sin(2k\pi+a)]\}=0.$$

Now each of these factors may be resolved into n simple factors. Thus we get, from the first and the second, respectively, the factors

$$x-\left(\cos\frac{2k\pi+a}{n}+i\sin\frac{2k\pi+a}{n}\right),$$

and

$$x-\left(\cos\frac{2k\pi+a}{n}-i\sin\frac{2k\pi+a}{n}\right);$$

the product of which is the quadratic factor

$$x^2-2\cos\frac{2k\pi+a}{n}.x+1,$$

which, by putting $k = 0, 1, 2 \ldots n-1$, respectively, gives all the quadratic factors of $x^{2n} - 2\cos\alpha \,.\, x^n + 1$. Hence

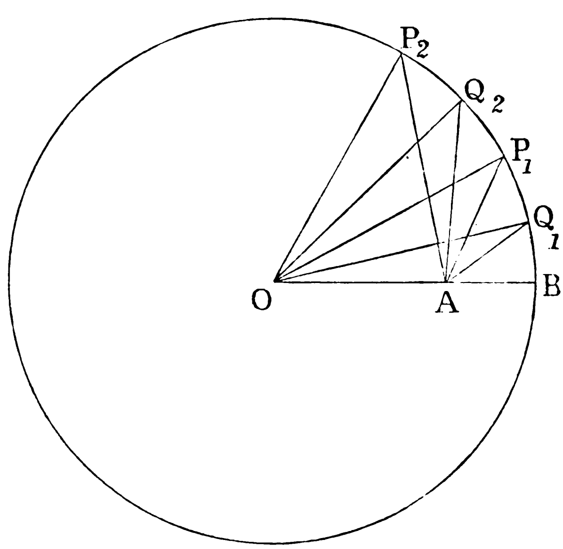
$$x^{2n} - 2\cos\alpha \,.\, x^n + 1$$
$$= \left(x^2 - 2\cos\frac{\alpha}{n} \,.\, x + 1\right)\left(x^2 - 2\cos\frac{2\pi+\alpha}{n}\, x + 1\right) \ldots$$
$$\left(x^2 - 2\cos\frac{2(n-1)\pi+\alpha}{n} \,.\, x + 1\right). \quad (540)$$

Cor.—In equation (540) change x into $\frac{x}{a}$ and clear of fractions, and we get

$$x^{2n} - 2\cos\alpha\, a^n x^n + a^{2n}$$
$$= \left(x^2 - 2\cos\frac{\alpha}{n} \,.\, ax + a^2\right)\left(x^2 - 2\cos\frac{2\pi+\alpha}{n} \,.\, ax + a^2\right) \ldots$$
$$\left(x^2 - 2\cos\frac{2(n-1)\pi+\alpha}{n} \,.\, ax + a^2\right). \quad (541)$$

172. Cotes's Theorem.

Let *OB be the radius of a circle whose centre is O; A any point in OB or in OB produced; then, if the circumference be divided into n equal parts* BP_1, P_1P_2, P_2P_3, *&c.; and into 2n equal parts*



BQ_1, Q_1P_1, P_1Q_2, *&c., the product* $AP_1 \,.\, AP_2 \,.\, AP_3 \ldots$ *n factors* $= \pm\,(OA^n - OB^n)$, *according as A is outside or inside the circle; and* $AQ_1 \,.\, AQ_2 \,.\, AQ_3 \ldots$ *n factors* $= OA^n + OB^n$.

DEM.—Let $OA = x$, $OB = a$; then we have

$$AP_1^2 = OA^2 - 2OA \cdot OP_1 \cos\frac{2\pi}{n} + a^2, \text{ \&c.};$$

$$\therefore \quad AP_1^2 \cdot AP_2^2 \cdot AP_3^2 \ldots n \text{ factors}$$

$$= \left(x^2 - 2ax \cdot \cos\frac{2\pi}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{4\pi}{n} + a^2\right)\ldots = (x^n - a^n)^2$$

from (541), by putting $\alpha = 0$. Hence

$$AP_1 \cdot AP_2 \cdot AP_3 \ldots n \text{ factors} = \pm(x^n - a^n). \qquad (542)$$

Again,

$$AQ_1 \cdot AP_1 \cdot AQ_2 \cdot AP_2 \ldots 2n \text{ factors} \pm (x^{2n} - a^{2n}).$$

Hence, by division,

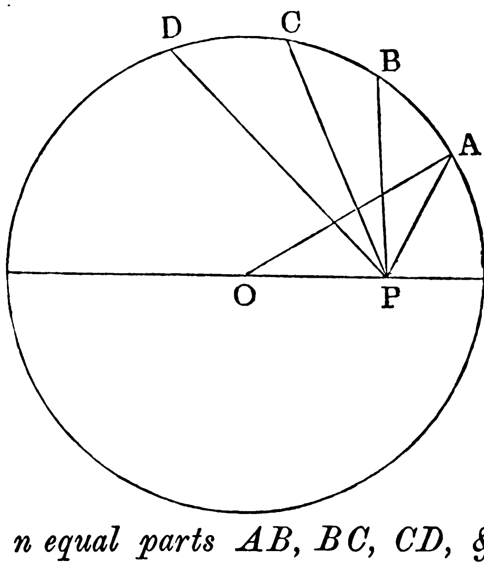
$$AQ_1 \cdot AQ_2 \cdot AQ_3 \ldots n \text{ factors} = x^n + a^n = OA^n + OB^n. \qquad (543)$$

Cor.—If the arcs BP_1, P_1P_2, &c., be trisected in the points V_1, V_1'; V_2, V_2', &c.; then

$$AV_1 \cdot AV_1' \cdot AV_2 \cdot AV_2', \text{ \&c., to } 2n \text{ factors}$$
$$= OA^{2n} + OA^n \cdot OB^n + OB^{2n}. \qquad (544)$$

173. De Moivre's Property of the Circle.

Let O be the centre, P any point in its plane. Divide the cir-



cumference into n equal parts AB, BC, CD, &c.; then, if the angle $POA = \theta$, *the product* $AP^2 \cdot BP^2 \cdot CP^2 \ldots n$ *factors*

$$= OA^{2n} - 2OA^n \cdot OP^n \cdot \cos n\theta + OP^{2n}. \qquad (545)$$

DEM.—Put $OA = a$, $OP = x$, $\theta = \dfrac{a}{n}$, and the theorem follows at once from equation (541).

EXERCISES.—XXXVI.

1. Extract the square root of $\cos 6\alpha + i \sin 6\alpha$.

2. ,, cube root of i.

3. Find the three cube roots of $3 + 4i$.

4. If $\alpha = \dfrac{\pi}{15}$, prove that $\cos\alpha \,.\, \cos 2\alpha \,.\, \cos 3\alpha \ldots \cos 7\alpha = 2^{-7}$. (546)

5. If $\alpha = \dfrac{2\pi}{17}$, prove that

$2(\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha)$ and $2(\cos 3\alpha + \cos 5\alpha + \cos 6\alpha + \cos 7\alpha)$ are the roots of the quadratic $x^2 + x - 4 = 0$. (547)

[Form the sum and the product of $2(\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha)$ and $2(\cos 3\alpha + \cos 5\alpha + \cos 6\alpha + \cos 7\alpha)$.]

6. If n be the product of two prime numbers, p, q, prove that all the roots of $x^n - 1 = 0$ will be obtained by multiplying the p roots of $x^p - 1 = 0$ by the q roots of $x^q - 1 = 0$.

7. Prove that the equation $x^{2n+1} = 1$ can be depressed to the n^{th} degree, and that the roots of the reduced equation are $2\cos\alpha$, $2\cos 2\alpha \ldots 2\cos n\alpha$, where

$$\alpha = \frac{2\pi}{2n+1}.$$

8. If a_1, a_2 be the roots of the equation (547); prove that

$2(\cos\alpha + \cos 4\alpha)$ and $2(\cos 2\alpha + \cos 8\alpha)$ are the roots of $x^2 - a_1 x - 1 = 0$, and (548)

$2(\cos 3\alpha + \cos 5\alpha)$, and $2(\cos 6\alpha + \cos 7\alpha)$ are the roots of $x^2 - a_2 x - 1 = 0$. (549)

9. If b_1, b_2 be the roots of (548), and c_1, c_2 of (549), prove—

1°. That $2\cos\alpha$, $2\cos 4\alpha$ are the roots of $x^2 - b_1 x + b_2 = 0$. (550)

2°. ,, $2\cos 2\alpha$, $2\cos 8\alpha$,, ,, $x^2 - b_2 x + c_1 = 0$. (551)

3°. ,, $2\cos 3\alpha$, $2\cos 5\alpha$,, ,, $x^2 - c_1 x + c_2 = 0$. (552)

4°. ,, $2\cos 6\alpha$, $2\cos 7\alpha$,, ,, $x^2 - c_2 x + b_1 = 0$. (553)

10. Prove the following relations between b_1, b_2; c_1, c_2, viz.—

$$b_2 = \frac{b_1 - 1}{b_1 + 1}, \quad c_1 = \frac{b_2 - 1}{b_2 + 1}, \quad c_2 = \frac{c_1 - 1}{c_1 + 1}, \quad b_1 = \frac{c_2 - 1}{c_2 + 1}. \qquad (554)$$

11. If $\alpha = \frac{2\pi}{15}$, form the quartic whose roots are $2\cos\alpha$, $2\cos 2\alpha$, $2\cos 4\alpha$, $2\cos 7\alpha$.

Section III.—Decomposition of Circular Functions.

174. From the equation (540), we get several important results:—

1°. Put $x = 1$, and we get

$$2(1 - \cos\alpha) = 2^n\left(1 - \cos\frac{\alpha}{n}\right)\left(1 - \cos\frac{2\pi + \alpha}{n}\right)\left(1 - \cos\frac{4\pi + \alpha}{n}\right)$$
$$\ldots\left(1 - \cos\frac{2(n-1)\pi + \alpha}{n}\right).$$

In this put $\alpha = 2n\theta$, and extract the square root, and we have

$$\sin n\theta = 2^{n-1}\sin\theta \,.\, \sin\left(\theta + \frac{\pi}{n}\right).\sin\left(\theta + \frac{2\pi}{n}\right).\sin\left(\theta + \frac{3\pi}{n}\right)$$
$$\ldots\sin\left(\theta + \frac{(n-1\pi)}{n}\right). \qquad (555)$$

2°. In (555) change θ into $\theta + \frac{\pi}{2n}$, and we get

$$\cos n\theta = 2^{n-1}\sin\left(\theta + \frac{\pi}{2n}\right)\sin\left(\theta + \frac{3\pi}{2n}\right)\ldots\sin\left(\theta + \frac{(2n-1)\pi}{2n}\right). \qquad (556)$$

3°. In (556) put $\theta = 0$, and we get

$$1 = 2^{n-1}\sin\frac{\pi}{2n}\,.\,\sin\frac{3\pi}{2n}\,.\,\sin\frac{5\pi}{2n}\ldots\sin\frac{(2n-1)\pi}{2n}. \qquad (557)$$

4°. The last factor in (555) is equal to

$$\sin\left(\frac{\pi}{n} - \theta\right);$$

the last but one equal to

$$\sin\left(\frac{2\pi}{n} - \theta\right), \text{ \&c.};$$

and if n be even, there is a factor equal to $\cos\theta$. Hence, for n even we have

$$\sin n\theta = 2^{n-2}\sin 2\theta\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)$$

$$\ldots\left(\sin^2\frac{(n-2)\pi}{n} - \sin^2\theta\right), \quad (559)$$

and if n be odd,

$$\sin n\theta = 2^{n-1}\sin\theta\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)$$

$$\ldots\left(\sin^2\frac{(n-1)\pi}{2n} - \sin^2\theta\right). \quad (560)$$

5°. In like manner, from (556), we get for n even

$$\cos n\theta = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)$$

$$\ldots\left(\sin^2\frac{(n-1)\pi}{2n} - \sin^2\theta\right), \quad (561)$$

and for n odd,

$$\cos n\theta = 2^{n-1}\cos\theta\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)$$

$$\left(\sin^2\frac{(n-2)\pi}{2n} - \sin^2\theta\right) \quad (562)$$

175. *To resolve $\sin\theta$ into factors.*

In (560) change θ into $\frac{\theta}{n}$, and divide both sides by $\sin\frac{\theta}{n}$. We get

$$\sin\theta \div \sin\frac{\theta}{n} = 2^{n-1}\left(\sin^2\frac{\pi}{n} - \sin^2\frac{\theta}{n}\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\frac{\theta}{n}\right)$$

$$\ldots\left(\sin^2\frac{(n-1)\pi}{2n} - \sin^2\frac{\theta}{n}\right). \quad (563)$$

Now, if θ diminish indefinitely, we get, since the limit of $\sin\theta \div \sin\frac{\theta}{n}$ is n, and the number of factors on the right is finite,

$$n = 2^{n-1}\sin^2\frac{\pi}{n}\cdot\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{(n-1)\pi}{2n}. \quad (564)$$

Hence, from (563), we get for n odd,

$$\sin\theta = n\sin\frac{\theta}{n}\left(1-\frac{\sin^2\theta/n}{\sin^2 2\pi/2n}\right)\left(1-\frac{\sin^2\theta/n}{\sin^2 4\pi/2n}\right)\ldots\left(1-\frac{\sin^2\theta/n}{\sin^2(n-1)\pi/2n}\right). \quad (565)$$

In like manner, we get from (559), for n even,

$$\sin\theta = n\sin\frac{\theta}{n}\cos\frac{\theta}{n}\left(1-\frac{\sin^2\theta/n}{\sin^2 2\pi/2n}\right)\ldots\left(1-\frac{\sin^2\theta/n}{\sin^2(n-2)\pi/2n}\right). \quad (566)$$

The formulae (565), (566) can be transformed by the identity

$$\left(1-\frac{\sin^2\alpha}{\sin^2\beta}\right) = \cos^2\alpha\left(1-\frac{\tan^2\alpha}{\tan^2\beta}\right)$$

into the following. Thus, n odd—

$$\sin\theta = \cos^n\frac{\theta}{n}\cdot n\tan\frac{\theta}{n}\left(1-\frac{\tan^2\theta/n}{\tan^2 2\pi/2n}\right)\ldots\left(1-\frac{\tan^2\theta/n}{\tan^2(n-1)\pi/2n}\right).$$

n even— (567)

$$\sin\theta = \cos^n\frac{\theta}{n}\cdot n\tan\frac{\theta}{n}\left(1-\frac{\tan^2\theta/n}{\tan^2 2\pi/2n}\right)\ldots\left(1-\frac{\tan^2\theta/n}{\tan^2(n-2)\pi/2n}\right). \quad (568)$$

Similarly, $\cos\theta$ may be resolved. Thus, from (561), (562), we get, n even—

$$\cos\theta = \left(1-\frac{\sin^2\theta/n}{\sin^2\pi/2n}\right)\left(1-\frac{\sin^2\theta/n}{\sin^2 3\pi/2n}\right)\ldots\left(1-\frac{\sin^2\theta/n}{\sin^2(n-1)\pi/2n}\right).$$

n odd— (569)

$$\cos\theta = \cos\frac{\theta}{n}\left(1-\frac{\sin^2\theta/n}{\sin^2\pi/2n}\right)\left(1-\frac{\sin^2\theta/n}{\sin^2 3\pi/2n}\right)\ldots\left(1-\frac{\sin^2\theta/n}{\sin^2(n-2)\pi/2n}\right). \quad (570)$$

And, transforming, by the identity,

$$1 - \frac{\sin^2 \alpha}{\sin^2 \beta} = \cos^2 \alpha \left(1 - \frac{\tan^2 \alpha}{\tan^2 \beta}\right),$$

these become, n even—

$$\cos\theta = \cos^n \frac{\theta}{n}\left(1 - \frac{\tan^2\theta/n}{\tan^2\pi/2n}\right)\left(1 - \frac{\tan^2\theta/n}{\tan^2 3\pi/2n}\right)\ldots\left(1 - \frac{\tan^2\theta/n}{\tan^2(n-1)\pi/2n}\right).$$

n odd— (571)

$$\cos\theta = \cos^n \frac{\theta}{n}\left(1 - \frac{\tan^2\theta/n}{\tan^2\pi/2n}\right)\left(1 - \frac{\tan^2\theta/n}{\tan^2 3\pi/2n}\right)\ldots\left(1 - \frac{\tan^2\theta/n}{\tan^2(n-2)\pi/2n}\right). \quad (572)$$

176. *To resolve sin θ into an infinite number of factors.*

Lemma.—*If a, b, be two angles, so that $a < b < \frac{\pi}{2}$, then*

$$1 - \frac{a^2}{b^2} > 1 - \frac{\sin^2 a}{\sin^2 b}, \quad \text{and} \quad < 1 - \frac{\tan^2 a}{\tan^2 b}.$$

This is easily proved.

Now, if $\theta < \pi$, since

$$\cos\frac{\theta}{n} < 1, \quad n\sin\frac{\theta}{n} < \theta, \quad \text{and } n\tan\frac{\theta}{n} > \theta,$$

we have, from equations (565), (568), by the lemma,

$$\sin\theta < \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots\left(1 - \frac{\theta^2}{(n-2)^2\pi^2}\right), \quad (1)$$

$$\sin\theta > \cos^n\frac{\theta}{n}\,.\,\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots\left(1 - \frac{\theta^2}{(n-2)^2}\pi^2\right). \quad (2)$$

Now, if $\cos^n\frac{\theta}{n}$ be denoted by $(1 - \epsilon_n)$, ϵ_n vanishes when n becomes infinite. Hence, it follows from the inequalities (1), (2), that

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots(1 - \eta_n),$$

where $\eta_n < \epsilon_n$. Hence, when n becomes infinite,

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots \text{ to infinity.} \quad (573)$$

Similarly,

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\ldots \text{ to infinity.} \quad (574)$$

177. From (573) and (489) we get

$$e^{i\theta} - e^{-i\theta} = 2i\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots$$

In this identity change θ into $i\theta$, and we get, after changing signs,

$$e^{\theta} - e^{-\theta} = 2\theta\left(1 + \frac{\theta^2}{\pi^2}\right)\left(1 + \frac{\theta^2}{2^2\pi^2}\right)\left(1 + \frac{\theta^2}{3^2\pi^2}\right)\ldots \text{ to infinity.} \quad (575)$$

Similarly, from (574) and (488), we get

$$e^{\theta} + e^{-\theta} = 2\left(1 + \frac{4\theta^2}{\pi^2}\right)\left(1 + \frac{4\theta^2}{3^2\pi^2}\right)\left(1 + \frac{4\theta^2}{5^2\pi^2}\right)\ldots \text{ to infinity.} \quad (576)$$

178. *To resolve* $\tan\theta$ *into fractions.*

By taking the logarithmic differential of (539), we get, for n even,

$$\frac{nx^{n-1}}{x^n + 1} = \frac{2x - 2\cos\pi/n}{x^2 - 2x\cos\pi/n + 1} + \frac{2x - 2\cos 3\pi/n}{x^2 - 2x\cos 3\pi/n + 1}$$
$$\ldots \frac{2x - 2\cos(n-1)\pi/n}{x^2 - 2x\cos(n-1)\pi/n + 1}.$$

In this identity change x into $\frac{x}{r}$; multiply the result by x, and from the product subtract the identity got by interchanging x and r, and we have

$$\frac{n(x^n - r^n)}{x^n + r^n} = 2\left\{\frac{x^2 - r^2}{x^2 - 2rx\cos\pi/n + r^2} + \frac{x^2 - r^2}{x^2 - 2rx\cos 3\pi/n + r^2}\right.$$
$$\left.\ldots \frac{x^2 - r^2}{x^2 - 2rx\cos(n-1)\pi/n + r^2}\right\}.$$

Substitute the values $e^{\frac{i\theta}{n}}$, $e^{-\frac{i\theta}{n}}$ for x, r, and we have, for n even,

$$\tan\theta = \frac{2}{n}\cot\frac{\theta}{n}\left\{\frac{\sin^2\theta/n}{\sin^2\pi/2n - \sin^2\theta/n} + \frac{\sin^2\theta/n}{\sin^2 3\pi/2n - \sin^2\theta/n}\right.$$
$$\left.\ldots\frac{\sin^2\theta/n}{\sin^2(n-1)\pi/2n - \sin^2\theta/n}\right\}. \quad (577)$$

Similarly, from (537), we get for n odd—

$$\tan\theta = \frac{1}{n}\tan\frac{\theta}{n} + \frac{2}{n}\cot\frac{\theta}{n}\left\{\frac{\sin^2\theta/n}{\sin^2\pi/2n - \sin^2\theta/n} + \frac{\sin^2\theta/n}{\sin^2 3\pi/2n - \sin^2\theta/n}\right.$$
$$\left.\ldots\frac{\sin^2\theta/n}{\sin^2(n-2)\pi/2n - \sin^2\theta/n}\right\}. \quad (578)$$

In like manner, from the equations (538), (536), we get, for n even—

$$\cot\theta = \frac{2}{n}\cot\frac{2\theta}{n} - \frac{2}{n}\cot\frac{\theta}{n}\left\{\frac{\sin^2\theta/n}{\sin^2 2\pi/2n - \sin^2\theta/n} + \frac{\sin^2\theta/n}{\sin^2 4\pi/2n - \sin^2\theta/n}\right.$$
$$\left.\ldots\frac{\sin^2\theta/n}{\sin^2(n-2)\pi/2n - \sin^2\theta/n}\right\}. \quad (579)$$

For n odd—

$$\cot\theta = \frac{1}{n}\cot\frac{\theta}{n} - \frac{2}{n}\cot\frac{\theta}{n}\left\{\frac{\sin^2\theta/n}{\sin^2 2\pi/2n - \sin^2\theta/n} + \frac{\sin^2\theta/n}{\sin^2 4\pi/2n - \sin^2\theta/n}\right.$$
$$\left.\ldots\frac{\sin^2\theta/n}{\sin^2(n-1)\pi/2n - \sin^2\theta/n}\right\}. \quad (580)$$

179. If we transform the equations (577)–(580) by the identity

$$\frac{\sin^2 A}{\sin^2 B - \sin^2 A} = \tan^2 A + \frac{\tan^2 A}{\cos^2 A(\tan^2 B - \tan^2 A)},$$

we get the four following equations:—

For n even—

$$\tan\theta = \tan\frac{\theta}{n} + \frac{2}{n}\left(\tan\frac{\theta}{n} + \cot\frac{\theta}{n}\right)\left\{\frac{\tan^2\theta/n}{\tan^2\pi/2n - \tan^2\theta/n}\right.$$
$$\left.+\frac{\tan^2\theta/n}{\tan^2 3\pi/2n - \tan^2\theta/n}\ldots\frac{\tan^2\theta/n}{\tan^2(n-1)\pi/2n - \tan^2\theta/n}\right\}. \quad (581)$$

For n odd—

$$\tan\theta = \tan\frac{\theta}{n} + \frac{2}{n}\left(\tan\frac{\theta}{n} + \cot\frac{\theta}{n}\right)\left\{\frac{\tan^2\theta/n}{\tan^2\pi/2n - \tan^2\theta/n}\right.$$

$$\left. + \frac{\tan^2\theta/n}{\tan^2 3\pi/2n - \tan^2\theta/n} \cdots \frac{\tan^2\theta/n}{\tan^2(n-2)\pi/2n - \tan^2\theta/n}\right\}. \quad (582)$$

For n even—

$$\cot\theta = \left(\frac{2}{n}\cot\frac{2\theta}{n} - \tan\frac{\theta}{n}\right) - \frac{2}{n}\left(\tan\frac{\theta}{n} + \cot\frac{\theta}{n}\right)$$

$$\left\{\frac{\tan^2\theta/n}{\tan^2 2\pi/2n - \tan^2\theta/n} \cdots \frac{\tan^2\theta/n}{\tan^2(n-2)\pi/2n - \tan^2\theta/n}\right\}. \quad (583)$$

For n odd—

$$\cot\theta = \frac{1}{n}\left(\tan\frac{\theta}{n} + \cot\frac{\theta}{n}\right) - \tan\frac{\theta}{n} - \frac{2}{n}\left(\tan\frac{\theta}{n} + \cot\frac{\theta}{n}\right)$$

$$\left\{\frac{\tan^2\theta/n}{\tan^2 2\pi/2n - \tan^2\theta/n} \cdots \frac{\tan^2\theta/n}{\tan^2(n-1)\pi/2n - \tan^2\theta/n}\right\}. \quad (584)$$

180. *Resolution of $\tan\theta$, $\cot\theta$, into an infinite number of simple fractions.*

If we take the logarithmic differential of equation (574), we get

$$\tan\theta = \frac{2\theta}{(\pi/2)^2 - \theta^2} + \frac{2\theta}{(3\pi/2)^2 - \theta^2} + \frac{2\theta}{(5\pi/2)^2 - \theta^2} + \text{\&c. to inf.,} \quad (585)$$

or
$$\tan\theta = \frac{1}{\pi/2 - \theta} - \frac{1}{\pi/2 + \theta} + \frac{1}{3\pi/2 - \theta} - \frac{1}{3\pi/2 + \theta} + \text{\&c.} \quad (586)$$

Similarly, from equation (573), we have

$$\cot\theta = \frac{1}{\theta} - \left\{\frac{2\theta}{\pi^2 - \theta^2} + \frac{2\theta}{(2\pi)^2 - \theta^2} + \frac{2\theta}{(3\pi)^2 - \theta^2} + \text{\&c.}\right\}, \quad (587)$$

or
$$\cot\theta = \frac{1}{\theta} - \left(\frac{1}{\pi - \theta} - \frac{1}{\pi + \theta}\right) - \left(\frac{1}{2\pi - \theta} - \frac{1}{2\pi + \theta}\right) - \text{\&c.} \quad (588)$$

181. *Resolution of $\operatorname{cosec}\theta$ and $\sec\theta$ into an infinite number of simple fractions.*

Since $$\operatorname{cosec}\theta = \tfrac{1}{2}(\tan\tfrac{1}{2}\theta + \cot\tfrac{1}{2}\theta),$$

we get, by changing θ into $\frac{1}{2}\theta$ in equations (586), (588), adding, dividing by 2, and reducing,

$$\operatorname{cosec}\theta = \frac{1}{\theta} + \left(\frac{1}{\pi-\theta} - \frac{1}{\pi+\theta}\right) - \left(\frac{1}{2\pi-\theta} - \frac{1}{2\pi+\theta}\right)$$

$$+ \left(\frac{1}{3\pi-\theta} - \frac{1}{3\pi+\theta}\right) - \&c., \tag{589}$$

or

$$\operatorname{cosec}\theta = \frac{1}{\theta} + \frac{2\theta}{\pi^2-\theta^2} - \frac{2\theta}{(2\pi)^2-\theta^2} + \frac{2\theta}{(3\pi)^2-\theta^2}. \ \&c. \tag{590}$$

If in these equations we change θ into $\frac{\pi}{2} - \theta$, we get

$$\sec\theta = \left(\frac{1}{\pi/2-\theta} + \frac{1}{\pi/2+\theta}\right) - \left(\frac{1}{3\pi/2-\theta} + \frac{1}{3\pi/2+\theta}\right)$$

$$+ \left(\frac{1}{5\pi/2-\theta} + \frac{1}{(5\pi/2+\theta)}\right), \ \&c. \tag{591}$$

or

$$\sec\theta = \frac{\pi}{(\pi/2)^2-\theta^2} - \frac{3\pi}{(3\pi/2)^2-\theta^2} + \frac{5\pi}{(5\pi/2)^2-\theta^2} - \&c. \tag{592}$$

Exercises.—XXXVII.

1. Prove $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. \text{ to inf.} = \frac{\pi^2}{6}. \tag{593}$$

[Equate the coefficients of θ^3 in the expressions

$$\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \&c.,$$

and $$\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots,$$

each of which is equal to $\sin\theta$.]

2. Prove $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\text{ to inf.}=\frac{\pi^2}{8}. \qquad (594)$$

3. Prove that the series for $\tan\theta$, equation (585), is convergent for all values of θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

[Make $\theta=\frac{\pi}{2}$, and the sum of all the terms after the first becomes

$$\frac{1}{\pi}\left(\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4},\text{ \&c., to inf.}\right)=\frac{1}{\pi};$$

and when θ is $<\frac{\pi}{2}$, the sum of all the terms after the first is $<\frac{1}{\pi}$ Therefore, &c.]

4. Prove $$\frac{\pi^2}{6}=\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1},\text{ \&c. to inf.,}$$

the numerators consisting of the squares of all the prime numbers.—(Crofton). (595)

5. Prove the series for $\cot\theta$ (587) is convergent for values of θ between

$$-\frac{\pi}{2}\text{ and }\frac{\pi}{2}.$$

6. If $$S_{2n}=\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\text{ \&c. to inf.,}$$

$$S'_{2n}=\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\text{ \&c.,}$$

prove that $$\tan\theta=\frac{2^3S_2.\theta}{\pi^2}+\frac{2^5S_4.\theta^3}{\pi^4}+\frac{2^7S_6.\theta^5}{\pi^6}+\text{ \&c.,} \qquad (596)$$

and $$\cot\theta=\frac{1}{\theta}-2\left(\frac{S_2'.\theta}{\pi^2}+\frac{S_4'.\theta^3}{\pi^4}+\frac{S_6'.\theta^5}{\pi^6}+\text{ \&c.}\right) \qquad (597)$$

[Expand the partial fractions in (586), (588) into series.]

7. Prove that $$S_{2n}'=\frac{2^{2n-1}.\pi^{2n}.B^n}{\lfloor 2n}. \qquad (598)$$

[Compare the values of $\tan\theta$, (528), (596).]

8. If n be even, prove

$$\sin\frac{\pi}{2n}.\sin\frac{3\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}=\left(\frac{1}{2}\right)^{\frac{n-1}{2}}. \qquad (599)$$

9. If n be even, prove

$$\cos\frac{2\pi}{2n}\cdot\cos\frac{4\pi}{2n}\ldots\cos\frac{(n-2)\pi}{2n} = \left(\frac{n}{2^{n-1}}\right)^{\frac{1}{2}}. \qquad (600)$$

10. Prove

$$\frac{\sin(\alpha-\theta)}{\sin\alpha} = \left(1-\frac{\theta}{\alpha}\right)\left(1+\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{\pi+\alpha}\right)\left(1+\frac{\theta}{2\pi-\alpha}\right)\left(1-\frac{\theta}{2\pi+\alpha}\right), \text{ \&c.} \qquad (601)$$

11. Prove

$$\frac{\sin(\alpha+\theta)}{\sin\alpha} = \left(1+\frac{\theta}{\alpha}\right)\left(1-\frac{\theta}{\pi-\alpha}\right)\left(1+\frac{\theta}{\pi+\alpha}\right)\left(1-\frac{\theta}{2\pi-\alpha}\right)\left(1+\frac{\theta}{2\pi+\alpha}\right)\ldots \qquad (602)$$

12. If n be an even integer,

$$\operatorname{cosec}^2\frac{\pi}{2n}+\operatorname{cosec}^2\frac{3\pi}{2n}+\operatorname{cosec}^2\frac{5\pi}{2n}\ldots\operatorname{cosec}^2\frac{(n-1)\pi}{2n} = \frac{n^2}{2}. \qquad (603)$$

13. Prove $\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\cdot\sin\frac{\pi+\theta}{n}\cdot\sin\frac{2\pi+\theta}{n}\ldots n$ factors. (604)

14. ,, n even, $\pm\sin\theta = 2^{n-1}\cos\frac{\theta}{n}\cdot\cos\frac{\pi+\theta}{n}\cdot\cos\frac{2\pi+\theta}{n}\ldots n$ factors. (605)

15. ,, n odd, $\pm\cos\theta = 2^{n-1}\cos\frac{\theta}{n}\cdot\cos\frac{\pi+\theta}{n}\cdot\cos\frac{2\pi+\theta}{n}\ldots n$ factors. (606)

16. ,, n even, $\pm 1 = \tan\frac{\theta}{n}\cdot\tan\frac{\pi+\theta}{n}\cdot\tan\frac{2\pi+\theta}{n}\ldots n$ factors. (607)

17. ,, n odd, $\tan\theta = \tan\frac{\theta}{n}\cdot\tan\frac{\pi+\theta}{n}\cdot\tan\frac{2\pi+\theta}{n}\ldots n$ factors. (608)

18. ,, Wallis's theorem, $\frac{\pi}{2} = \frac{2^2}{2^2-1}\cdot\frac{4^2}{4^2-1}\cdot\frac{6^2}{6^2-1}\ldots$ to inf. (609)

19. ,, $1^{-4}+2^{-4}+3^{-4}+$ &c. to inf. $=\frac{\pi^4}{90}$. (610)

20. ,, $1^{-6}+2^{-6}+3^{-6}+$ &c. to inf. $=\frac{\pi^6}{945}$. (611)

21. ,, $1^{-8}+2^{-8}+3^{-8}+$ &c. to inf. $=\frac{\pi^8}{9450}$. (612)

22. ,, $1^{-10}+2^{-10}+3^{-10}+$ &c. to inf. $=\frac{\pi^{10}}{93555}$ (613)

23. ,, $\frac{\pi^4}{90} = \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\ldots$ to inf. (614)

24. Prove $\dfrac{\pi^6}{945} = \dfrac{2^6}{2^6-1} \cdot \dfrac{3^6}{3^6-1} \cdot \dfrac{5^6}{5^6-1} \cdot \dfrac{7^6}{7^6-1} \cdot \dfrac{11^6}{11^6-1} \ldots$ to inf. (615)

25. ,, $\dfrac{\pi^8}{9450} = \dfrac{2^8}{2^8-1} \cdot \dfrac{3^8}{3^8-1} \cdot \dfrac{5^8}{5^8-1} \cdot \dfrac{7^8}{7^8-1} \cdot \dfrac{11^8}{11^8-1} \ldots$ to inf. (616)

26. ,, $\dfrac{\pi^2}{15} = \dfrac{2^2}{2^2+1} \cdot \dfrac{3^2}{3^2+1} \cdot \dfrac{5^2}{5^2+1} \cdot \dfrac{7^2}{7^2+1} \cdot \dfrac{11^2}{11^2+1} \ldots$ to inf. (617)

27. ,, $\dfrac{\pi^4}{105} = \dfrac{2^4}{2^4+1} \cdot \dfrac{3^4}{3^4+1} \cdot \dfrac{5^4}{5^4+1} \cdot \dfrac{7^4}{7^4+1} \cdot \dfrac{11^4}{11^4+1} \ldots$ to inf. (618)

28. ,, $\sqrt{2} = \dfrac{2^2}{2^2-1} \cdot \dfrac{6^2}{6^2-1} \cdot \dfrac{10^2}{10^2-1} \cdot \dfrac{14^2}{14^2-1} \ldots$ to inf. (619)

29. ,, $\dfrac{\sqrt{3}}{2} = \dfrac{3^2-1}{3^2} \cdot \dfrac{9^2-1}{9^2} \cdot \dfrac{15^2-1}{15^2} \ldots$ to inf. (620)

30. Prove $\sin x + \cos x =$

$$\sqrt{2}\left\{\left(x+\frac{\pi}{4}\right)\left(1-\left(\frac{4x+\pi}{4\pi}\right)^2\right)\left(1-\left(\frac{4x+\pi}{8\pi}\right)^2\right)\left(1-\left(\frac{4x+\pi}{12\pi}\right)^2\right)\ldots \text{to inf.}\right\} \quad (621)$$

31. Prove $\sin x - \cos x =$

$$\sqrt{2}\left\{\left(x-\frac{\pi}{4}\right)\left(1-\left(\frac{4x-\pi}{4\pi}\right)^2\right)\left(1-\left(\frac{4x-\pi}{8\pi}\right)^2\right)\left(1-\left(\frac{4x-\pi}{12\pi}\right)^2\right)\ldots \text{to inf.}\right\} \quad (622)$$

32. Prove, if $e^x \cos x$ be expanded in ascending powers of x, that the coefficient of x^n

$$= \underline{|n}\left\{1 - \frac{\underline{|n}}{\underline{|2}\,\underline{|n-2}} + \frac{\underline{|n}}{\underline{|4}\,\underline{|n-4}} - \&c.\right\},$$

and also $$= 2^{\frac{n}{2}} \cos n\pi/4. \quad (623)$$

33. Prove $\dfrac{\cos\theta + \cos\phi}{1+\cos\phi} = \left\{1 - \dfrac{\theta^2}{(\pi-\phi)^2}\right\}\left\{1 - \dfrac{\theta^2}{(\pi+\phi)^2}\right\}$

$$\left\{1 - \frac{\theta^2}{(3\pi-\phi)^2}\right\}\left\{1 - \frac{\theta^2}{(3\pi+\phi)^2}\right\} \ldots \text{to inf.} \quad (624)$$

34. Expand $(a^2 - 2ab\cos\phi + b^2)^{\frac{1}{2}}$ in a series of the form

$$A_0 + A_1 \cos\phi + A_2 \cos 2\phi + A_3 \cos 3\phi, \ \&c.$$

35. If
$$\sec\theta = 1 + \frac{A_2\theta^2}{\lfloor 2} + \frac{A_4\theta^4}{\lfloor 4} + \frac{A_6\theta^6}{\lfloor 6} + \&c.,$$
prove that
$$1 - \frac{\lfloor 2n}{\lfloor 2\lfloor 2n-2}A_2 + \frac{\lfloor 2n}{\lfloor 4\lfloor 2n-4}A_4 - \frac{\lfloor 2n}{\lfloor 6\lfloor 2n-6}A_6$$
$$+ \&c. \ldots (-1)^n A_{2n} = 0. \quad (625)$$

36. Prove
$$\frac{\cos\theta-\cos\phi}{1-\cos\phi} = \left\{1-\frac{\theta^2}{\phi^2}\right\}\left\{1-\frac{\theta^2}{(2\pi-\phi)^2}\right\}\left\{1-\frac{\theta^2}{(2\pi+\phi)^2}\right\}$$
$$\left\{1-\frac{\theta^2}{(4\pi-\phi)^2}\right\}\left\{1-\frac{\theta^2}{(4\pi+\phi)^2}\right\}, \&c. \quad (626)$$

37. ,, $$1^{-2} - 2^{-2} + 3^{-2} - 4^{-2} + \&c. \text{ to infinity} = \frac{\pi^2}{12}. \quad (627)$$

38. ,, $$1^{-4} - 2^{-4} + 3^{-4} - 4^{-4} + \&c. \text{ to inf.} = \frac{7\pi^4}{720}. \quad (628)$$

39. ,, $$1^{-6} - 2^{-6} + 3^{-6} - 4^{-6} + \&c. \text{ to inf.} = \frac{31\pi^6}{30240}. \quad (629)$$

40. ,, $$\frac{\pi^2}{7} = \frac{2^4+2^2}{2^4+1}\cdot\frac{3^4+3^2}{3^4+1}\cdot\frac{5^4+5^2}{5^4+1}\cdot\frac{7^4+7^2}{7^4+1}\cdot\frac{11^4+11^2}{11^4+1}, \&c. \quad (630)$$

41. Prove
$$e^x + e^y = 2e^{\frac{1}{2}(x+y)}\left\{1+\left(\frac{x-y}{\pi}\right)^2\right\}\left\{1+\left(\frac{x-y}{3\pi}\right)^2\right\}\left\{1+\left(\frac{x-y}{5\pi}\right)^2\right\}, \&c. \text{ to inf.} \quad (631)$$

42. Prove
$$e^x + e^{-y} = 2e^{\frac{1}{2}(x-y)}\left\{1+\left(\frac{x+y}{\pi}\right)^2\right\}\left\{1+\left(\frac{x+y}{3\pi}\right)^2\right\}\left\{1+\left(\frac{x-y}{5\pi}\right)^2\right\}, \&c. \text{ to inf.} \quad (632)$$

43. Prove $$\frac{2}{5} = \frac{2^2-1}{2^2+1}\cdot\frac{3^2-1}{3^2+1}\cdot\frac{5^2-1}{5^2+1}\cdot\frac{7^2-1}{7^2+1}\cdot\frac{11^2-1}{11^2+1}\ldots \text{ to inf.} \quad (633)$$

44. ,, $$x = \sin x + \tfrac{1}{2}\cdot\frac{\sin^3 x}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{\sin^5 x}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\sin^7 x}{7}, \&c. \text{ to inf.} \quad (634)$$

45. ,, $$\sec x = 1 + \tfrac{1}{2}\,.\,\sin^2 x + \frac{1\,.\,3}{2\,.\,4}\sin^4 x + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\sin^6 x, \&c. \text{ to inf.} \quad (635)$$

46. ,, $$x^2 = \sin^2 x + \tfrac{2}{3}\,.\,\frac{\sin^4 x}{2} + \frac{2\,.\,4}{3\,.\,5}\frac{\sin^6 x}{3} + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\frac{\sin^8 x}{4}, \&c. \quad (636)$$

47. Prove $\dfrac{1^2}{1^2+1}+\dfrac{2^2}{2^2+1}+\dfrac{3^2}{3^2+1}+\ldots \text{ to inf.} = \dfrac{2\pi}{e^{\pi}-e^{-\pi}}$. (637)

48. ,, $$\frac{\sin\theta+\sin\phi}{\sin\phi} = \left(1+\frac{\theta}{\phi}\right)\left(1+\frac{\theta}{\pi-\phi}\right)\left(1-\frac{\theta}{\pi+\phi}\right)$$
$$\left(1+\frac{\theta}{2\pi+\phi}\right)\left(1-\frac{\theta}{2\pi+\phi}\right),\ldots \quad (638)$$

49. ,, $$\frac{\sin\theta-\sin\phi}{\sin\phi} = -\left(1-\frac{\theta}{\phi}\right)\left(1+\frac{\theta}{\pi+\phi}\right)\left(1-\frac{\theta}{\pi-\phi}\right)$$
$$\left(1+\frac{\theta}{2\pi+\phi}\right)\left(1-\frac{\theta}{2\pi-\phi}\right),\ldots \quad (639)$$

50. ,, $$\cos\theta+\tan\tfrac{1}{2}\phi\,.\sin\theta = \left(1+\frac{2\theta}{\pi-\phi}\right)\left(1-\frac{2\theta}{\pi+\phi}\right)$$
$$\left(1+\frac{2\theta}{3\pi-\phi}\right)\left(1-\frac{2\theta}{3\pi+\phi}\right),\ldots \quad (640)$$

51. ,, $$\frac{\pi^2}{9} = \frac{2^6+2^2}{2^6-1}\cdot\frac{3^6+3^2}{3^6-1}\cdot\frac{5^6+5^2}{5^6-1}\cdot\frac{7^6+7^2}{7^6-1}\cdot\frac{11^6+11^2}{11^6-1},\ldots \quad (641)$$

Section IV.—Summation of Trigonometrical Series.

182. *To find the sum of a series of cosines of angles in AP.*

Let the proposed series be

$$\cos\alpha,\ \cos(\alpha+2\beta),\ \cos(\alpha+4\beta), \text{ \&c., to } n \text{ terms.}$$

Let $\cos\alpha+\cos(\alpha+2\beta)+\cos(\alpha+4\beta)\ldots\cos(\alpha+2(n-1)\beta) = S$. Multiply by $2\sin\beta$; decompose each product on the left into the difference between two sines, and we have

$$2S\,.\sin\beta = \sin\{\alpha+(2n-1)\beta\} - \sin(\alpha-\beta);$$

$$\therefore\quad S = \frac{\cos\{\alpha+(n-1)\beta\}\sin n\beta}{\sin\beta}. \quad (642)$$

From this formula we can infer other results. Thus:—

1°. Change α into $\left(\alpha-\dfrac{\pi}{2}\right)$, and we have the sum of the series $\sin\alpha,\ \sin(\alpha+2\beta),\ \sin(\alpha+4\beta)$, &c., to n terms,

$$= \frac{\sin(\alpha+(n-1)\beta)\sin n\beta}{\sin\beta}. \quad (643)$$

2°. In (642), (643), change β into $\beta + \frac{\pi}{2}$, and we have

$$\cos\alpha - \cos(\alpha + 2\beta) + \cos(\alpha + 4\beta)\ldots n \text{ terms}$$

$$= \frac{\cos\{\alpha + \overline{n-1}\,(\beta + \pi/2)\}\sin n(\beta + \pi/2)}{\cos\beta}. \quad (644)$$

$$\sin\alpha - \sin(\alpha + 2\beta) + \sin(\alpha + 4\beta)\ldots n \text{ terms}$$

$$= \frac{\sin\{\alpha + \overline{n-1}\,(\beta + \pi/2)\}\sin n(\beta + \pi/2)}{\cos\beta}. \quad (645)$$

3°. In (642), (643) put $\beta = \frac{\pi}{n}$, and we get

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cos\left(\alpha + \frac{4\pi}{n}\right) + \&c., \text{ to } n \text{ terms} = 0. \quad (646)$$

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \&c., \text{ to } n \text{ terms} = 0. \quad (647)$$

183. *Sum the series cosec* α + *cosec* 2α + *cosec* 4α, *&c., to* n *terms.*

We have, equation (142),

$$\operatorname{cosec}\alpha = \cot\tfrac{1}{2}\alpha - \cot\alpha.$$

Hence, changing α into 2α, 4α, &c., and adding, we get

$$S = \cot\tfrac{1}{2}\alpha - \cot 2^{n-1}\alpha. \quad (648)$$

Cor.—Hence the sum of the series

$$\tan\alpha + \cot\alpha, \quad \tan 2\alpha + \cot 2\alpha, \quad \tan 4\alpha + \cot 4\alpha, \ \&c.,$$

to n terms, may be found.

The artifice employed in this example of decomposing each term into the difference of two others is extensively used in the summation of series. As another example of the process, let it be required to find the sum of n terms of the series

$$\tfrac{1}{2}\tan\tfrac{1}{2}\alpha, \quad \frac{1}{2^2}\tan\frac{\alpha}{2^2}, \quad \frac{1}{2^3}\tan\frac{\alpha}{2^3}, \ \&c.$$

In this case the decomposition is

$$\tfrac{1}{2}\tan\tfrac{1}{2}a = \tfrac{1}{2}\cot\tfrac{1}{2}a - \cot a,$$

and changing a into $\frac{1}{2}a$, $\frac{1}{4}a$, &c., we find the sum

$$= \frac{1}{2^n}\cot\frac{a}{2^n} - \cot a. \qquad (649)$$

When the terms are not immediately reducible to differences which would cancel in addition, they may become so when multiplied by a factor. Thus, if the series be

$$\frac{1}{\cos a \,.\, \cos 3a} + \frac{1}{\cos 3a \,.\, \cos 5a} + \frac{1}{\cos 5a \,.\, \cos 7a}, \text{ \&c.,}$$

it becomes of the desired form if multiplied by $\sin 2a$, since

$$\frac{\sin 2a}{\cos a \,.\, \cos 3a} = \tan 3a - \tan a.$$

Hence the method of finding the sum is obvious.

184. Euler's theorem, § 160, viz.:—

$$\cos\theta + i\sin\theta = e^{i\theta},$$

and the theorems (§ 165),

$$2\cos\theta = e^{i\theta} + e^{-i\theta}, \quad 2i\sin\theta = e^{i\theta} - e^{-i\theta},$$

enable us to find by algebraic methods, such as the Binomial Theorem, the Exponential Theorem, Recurring Series, &c., the sums of extensive classes of progressions. Thus, to sum the series

$$\sin a + c\sin(a+\beta) + c^2\sin(a+2\beta) + \text{\&c., to } n \text{ terms.}$$

Let S denote the sum. Multiply by $2i$; then $2iS$ is equal to the difference between $2\,GP$'s, whose common ratios are, respectively, $ce^{i\beta}$ and $ce^{-i\beta}$. Hence

$$2iS = e^{ia}\left(\frac{1 - c^n e^{ni\beta}}{1 - ce^{i\beta}}\right) - e^{-ia}\left(\frac{1 - e^n e^{-ni\beta}}{1 - ce^{-i\beta}}\right);$$

therefore

$$S = \frac{\sin a - c\sin(a-\beta) - c^n \sin(a+n\beta) + c^{n+1}\sin(a+\overline{n-1}\beta)}{1 - 2c\cos\beta + c^2}. \tag{650}$$

This example may, like that in § 182, be made to give several results. Thus, if a be changed into $a + \frac{\pi}{2}$, we get the sum of the series

$$\cos a + c\cos(a+\beta) + c^2\cos(a+2\beta)\ldots \text{ to } n \text{ terms}$$

$$= \frac{\cos a - c\cos(a-\beta) - c^n\cos(a+n\beta) + c^{n+1}\cos(a+\overline{n-1}\beta)}{1-2c\cos\beta + c^2}. \tag{651}$$

These two progressions may be summed differently, as follows:—Thus, put S for the first, and C for the second; then we have $C + iS$

$$= (\cos a + i\sin a)\{1 + c(\cos\beta + i\sin\beta) + c^2(\cos 2\beta + i\sin 2\beta), \&c.\}$$

$$= e^{ia}(1 + ce^{i\beta} + c^2e^{2i\beta} + c^3e^{3i\beta} + \&c.) = e^{ia}\left(\frac{1-c^ne^{ni\beta}}{1-ce^{i\beta}}\right)$$

$$= e^{ia}(1 - ce^{-i\beta})\frac{1-c^n e^{in\beta})}{1-2c\cos\beta + c^2}.$$

Substitute for the exponentials their equivalents in sines and cosines, and then compare real and imaginary parts.

If c be less than unity, the two preceding series are convergent, and the sums to infinity are, respectively,

$$\frac{\sin a - c\sin(a-\beta)}{1-2c\cos\beta + c^2}, \quad \frac{\cos a - c\cos(a-\beta)}{1-2c\cos\beta+c^2}. \tag{652}$$

As another example, the sum of the series

$$\cos a + x\cos(a+\beta) + \frac{x^2}{\underline{|2}}\cos(a+2\beta) + \&c.,$$

is found by the exponential theorem to be

$$e^{x\cos\beta}\cos(a + x\sin\beta),$$

and the sum of the corresponding series with sines to be

$$e^{x\cos\beta}\sin(a + x\sin\beta).$$

185. The following general theorem includes the two preceding examples as particular cases:—

If the sum of the series

$$c_0 + c_1 x + c_2 x^2 \ldots c_n x^n = f(x),$$

then the sum of

$$c_0 \cos \alpha + c_1 \cos(\alpha + \beta) + c_2 \cos(\alpha + 2\beta)$$

$$\ldots c_n \cos(\alpha + n\beta) = \tfrac{1}{2}\{e^{i\alpha} f(e^{i\beta}) + e^{-i\alpha} f(e^{-i\beta})\}, \quad (653)$$

and of

$$c_0 \sin \alpha + c_1 \sin(\alpha + \beta) + c_2 \sin(\alpha + 2\beta)$$

$$\ldots c_n \sin(\alpha + n\beta) = \tfrac{1}{2}\{e^{i\alpha} f(e^{i\beta}) - e^{-i\alpha} f(e^{-i\beta})\}. \quad (654)$$

The proofs are evident by replacing the sines and cosines by their exponential values.

As an example, we find, putting

$$\tan \phi = \frac{h \sin \beta}{1 + h \cos \beta}, \quad r^2 = 1 + 2h \cos \beta + h^2,$$

the sum of n terms of the series

$$\cos \alpha + nh \cos(\alpha + \beta) + \frac{n \cdot n - 1}{\underline{|2}} h^2 \cos(\alpha + 2\beta) + \&c.,$$

is

$$r^n \cos(n\phi + \alpha), \quad (655)$$

and the sum of the corresponding series with sines

$$= r^n \sin(n\phi + \alpha). \quad (656)$$

These examples may also be solved by the second method in § 184. The methods explained in §§ 182, 183, being applications of the formulae for the transformations of the products of sines or cosines into sums or differences, belong to Chapter II., but we have preferred, however, not to give them there, in order that we may place together the various methods employed in the summation of trigonometrical series.

Exercises XXXVIII.

Find the sum of n terms of each of the following series:—

1. $\sec\alpha\,.\,\sec 2\alpha + \sec 2\alpha\,.\,\sec 3\alpha + \sec 3\alpha\,.\,\sec 4\alpha\ldots$

2. $\dfrac{1}{\cos\alpha+\cos 3\alpha}+\dfrac{1}{\cos\alpha+\cos 5\alpha}+\dfrac{1}{\cos\alpha+\cos 7\alpha}\ldots$

3. $\operatorname{cosec}\alpha\,.\,\sec 2\alpha - \sec 2\alpha\,.\,\operatorname{cosec} 3\alpha + \operatorname{cosec} 3\alpha\,.\,\sec 4\alpha\ldots$

4. $\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)\ldots$

5. $\sin^3\alpha+\sin^3(\alpha+\beta)+\sin^3(\alpha+2\beta)\ldots$

6. $\sin\alpha\,.\,\sin 2\alpha+\sin 2\alpha\,.\,\sin 3\alpha+\sin 3\alpha\,.\,\sin 4\alpha\ldots$

7. $\operatorname{cosec}\alpha\,.\,\operatorname{cosec}2\alpha+\operatorname{cosec}2\alpha\,.\,\operatorname{cosec}3\alpha+\operatorname{cosec}3\alpha\,.\,\operatorname{cosec}4\alpha\ldots$

8. $\operatorname{cosec}\alpha\,.\,\operatorname{cosec}2\alpha-\operatorname{cosec}2\alpha\,.\,\sec 3\alpha+\sec 3\alpha\,.\,\operatorname{cosec}4\alpha\ldots$

9. $\cos\alpha\,.\,\sin 2\alpha+\sin 2\alpha\,.\,\cos 3\alpha+\cos 3\alpha\,.\,\sin 4\alpha\ldots$

10. $\cos^4\alpha+\cos^4(\alpha+\beta)+\cos^4(\alpha+2\beta)\ldots$

11. $\sin\alpha\,.\,\sin 2\alpha+\sin 2\alpha\,.\,\sin 4\alpha+\sin 3\alpha\,.\,\sin 6\alpha\ldots$

12. $\dfrac{1}{1+\tan\alpha\,.\,\tan 2\alpha}+\dfrac{1}{1+\tan 2\alpha\,.\,\tan 4\alpha}+\dfrac{1}{1+\tan 3\alpha\,.\,\tan 6\alpha}\ldots$

13. $\sin\alpha-\sin 2\alpha+\sin 3\alpha-\sin 4\alpha\ldots$

14. $\cos\theta\,.\,\cos(\theta+\alpha)+\cos(\theta+\alpha)\,.\,\cos(\theta+2\alpha)+\cos(\theta+2\alpha)\,.\,(\cos\theta+3\alpha)\ldots$

15. $\dfrac{2\tan\theta}{1+\tan^2\theta}+\dfrac{2\tan 2\theta}{1+\tan^2 2\theta}+\dfrac{2\tan 3\theta}{1+\tan^2 3\theta}\ldots$

16. $\sec\alpha\,.\,\sec 3\alpha+\sec 2\alpha\,.\,\sec 4\alpha+\sec 3\alpha\,.\,\sec 5\alpha\ldots$

17. $\cos\alpha\,.\,\sin\beta+\cos 3\alpha\,.\,\sin 2\beta+\cos 5\alpha\,.\,\sin 3\beta\ldots$

18. $\sin\alpha+2^2\sin(\alpha+\beta)+3^2\sin(\alpha+2\beta)\ldots$

19. $\sin\alpha+2^3\sin(\alpha+\beta)+3^3\sin(\alpha+2\beta)\ldots$

20. $\dfrac{1}{2\cos^2\alpha/2}+\dfrac{1}{2^2\cos^2\alpha/2^2}+\dfrac{1}{2^3\cos^2\alpha/2^3}\ldots$

21. $\sec^2\alpha+2\sec^2 2\alpha+3\sec^2 4\alpha\ldots$

22. $\sin\alpha\,.\,\sin^2\frac{1}{2}\alpha+2\sin\frac{1}{2}\alpha\,.\,\sin^2\frac{1}{4}\alpha+4\sin\frac{1}{4}\alpha\,.\,\sin^2\frac{1}{8}\alpha\ldots$

23. $\tan\alpha\,.\,\sec^2\alpha+(\frac{1}{2}\tan\frac{1}{2}\alpha)(\frac{1}{2}\sec\frac{1}{2}\alpha)^2+(\frac{1}{4}\tan\frac{1}{4}\alpha)(\frac{1}{4}\sec\frac{1}{4}\alpha)^2\ldots$

24. $\sin^4\alpha+4\sin^4\dfrac{\alpha}{2}+4^2\sin^4\dfrac{\alpha}{2^2}+4^3\sin^4\dfrac{\alpha}{2^3}\ldots$

25. $\sin^5\alpha+\sin^5(\alpha+\beta)+\sin^5(\alpha+2\beta)\ldots$

26. $\sin\alpha + 2\sin 2\alpha + 3\sin 3\alpha + 4\sin 4\alpha \ldots$

27. $1 + n\sin\alpha + \dfrac{n\,.\,n-1}{\underline{|2}}\sin 2\alpha + \dfrac{n\,.\,n-1\,.\,n-2}{\underline{|3}}\sin 3\alpha.$

28. $\cos\alpha + 4\cos 2\alpha + 9\cos 3\alpha + 16\cos 4\alpha \ldots$

29. $x\sin\alpha - x^2\sin(\alpha+\beta) + x^3\sin(\alpha + 2\beta)\ldots$

30. $\cos^2\alpha + 2\cos^2 2\alpha + 3\cos^2 3\alpha + 4\cos^2 4\alpha.$

31. $\tan^{-1}x + \tan^{-1}\dfrac{x}{1+1\,.\,2x^2} + \tan^{-1}\dfrac{x}{1+2\,.\,3x^3}\ldots$

32. $\dfrac{\sin\theta}{\cos\theta + \cos 2\theta} + \dfrac{\sin 2\theta}{\cos\theta + \cos 4\theta} + \dfrac{\sin 3\theta}{\cos\theta + \cos 6\theta}\ldots$

Sum to infinity the following series:—

33. $\cos\alpha + \dfrac{\cos\alpha\,.\,\cos 2\alpha}{\underline{|1}} + \dfrac{\cos^2\alpha\,.\,\cos 3\alpha}{\underline{|2}} + \dfrac{\cos^3\alpha\,.\,\cos 4\alpha}{\underline{|3}}\ldots$

34. $\sin\alpha - \dfrac{\sin 2\alpha}{\underline{|2}} + \dfrac{\sin 3\alpha}{\underline{|3}} - \dfrac{\sin 4\alpha}{\underline{|4}}\ldots$

35. $1 - \dfrac{\cos 2\alpha}{\underline{|2}} + \dfrac{\cos 4\alpha}{\underline{|4}} - \dfrac{\cos 6\alpha}{\underline{|6}}\ldots$

36. $x\sin\alpha + \dfrac{x^2\sin 2\alpha}{\underline{|2}} + \dfrac{x^3\sin 3\alpha}{\underline{|3}} + \dfrac{x^4\sin 4\alpha}{\underline{|4}}\ldots$

37. $1 - \dfrac{\cos\alpha\cos\beta}{\underline{|1}} + \dfrac{\cos^2\alpha\,.\,\cos 2\beta}{\underline{|2}} - \dfrac{\cos^3\alpha\cos 3\beta}{\underline{|3}} + \&c.$

38. $\dfrac{\sin\alpha\cos\alpha}{\underline{|1}} + \dfrac{\sin 2\alpha\cos^2\alpha}{\underline{|2}} + \dfrac{\sin 3\alpha\cos^3\alpha}{\underline{|3}} + \&c.\ldots$

39. $\cos\alpha + \dfrac{\sin\alpha\cos 2\alpha}{\underline{|1}} + \dfrac{\sin^2\alpha\cos 3\alpha}{\underline{|2}} + \dfrac{\sin^3\alpha\cos 4\alpha}{\underline{|3}}, \&c.\ldots$

40. $\sin\alpha - \frac{1}{2}\sin 2\alpha + \frac{1}{3}\sin 3\alpha - \frac{1}{4}\sin 4\alpha, \&c.\ldots$

41. $\cos\alpha + \frac{1}{2}\cos 2\alpha + \quad \cos 3\alpha + \frac{1}{4}\cos 4\alpha, \&c.\ldots$

42. $\cos\alpha\cos\beta + \frac{1}{2}\cos^2\alpha\cos 2\beta + \frac{1}{3}\cos^3\alpha\cos 3\beta, \&c.\ldots$

43. $\sin\alpha\cos\beta - \frac{1}{2}\sin^2\alpha\cos 2\beta + \frac{1}{3}\sin^3\alpha\cos 3\beta, \&c.\ldots$

44. Prove $\cos\alpha - \frac{1}{2}\cos 2\alpha + \frac{1}{3}\cos 3\alpha - \&c. = \log(2\cos\frac{1}{2}\alpha).$ (657)

45. ,, $\cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \&c. = \frac{1}{2}\log\cot\frac{1}{2}\theta).$ (658)

46. ,, $\log\cos\dfrac{\theta}{2} + \log\cos\dfrac{\theta}{2^2} + \log\cos\dfrac{\theta}{2^3}\ldots = \log\sin\theta - \log\theta.$ (659)

Sum to infinity the following series, in each of which x is less than unity :—

47. $\cos\alpha + x\cos(\alpha+\beta) + \frac{x \,.\, x-1}{\lfloor 2}\cos(\alpha+2\beta)\ldots$

48. $\sin\alpha + x\sin(\alpha+\beta) + \frac{x(x-1)}{2}\sin(\alpha+2\beta)\ldots$

This and the preceding series are summed by the Binomial Theorem. Thus, denoting one by C, and the other by S, we have

$$C + iS \equiv e^{i\alpha}(1+e^{i\beta})^x = e^{i\alpha}\{(e^{\frac{1}{2}i\beta}+e^{-\frac{1}{2}i\beta})\,e^{\frac{1}{2}i\beta}\}^x$$

$$= (2\cos\tfrac{1}{2}\beta)^x . e^{i(\alpha+\frac{1}{2}x\beta)} = 2^x\cos^x\beta/2\,\{(\cos\alpha+\tfrac{1}{2}x\beta) + i\sin(\alpha+\tfrac{1}{2}x\beta)\}.$$

Hence $$C = 2^x\cos^x\beta/2 \,.\, \cos(\alpha+\tfrac{1}{2}x\beta),$$

$$S = 2^x\cos^x\beta/2 \,.\, \sin(\alpha+\tfrac{1}{2}x\beta).$$

49. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta\ldots$

[Make use of equation (500).]

50. $\sin\alpha + x\sin 2\alpha + \frac{x^2}{\lfloor 2}\sin 3\alpha\ldots$

51. $1 - x\cos\alpha + \frac{x^2\cos 2\alpha}{\lfloor 2} - \frac{x^3\cos 3\alpha}{\lfloor 3}\ldots$

52. $e^x\sin x - \frac{1}{2}e^{2x}\sin 2x + \frac{1}{3}e^{3x}\sin 3x\ldots$

53. $1 + \frac{x^2\sin 2\alpha}{\lfloor 2} + \frac{x^4\sin 4\alpha}{\lfloor 4}$, &c. . . .

54. $\sin\alpha + nx\sin(\alpha+\beta) + \frac{n \,.\, n-1}{\lfloor 2}x^2\sin(\alpha+2\beta)\ldots$ to $(n+1)$ terms.

55. $4 + 9x\cos\theta + 21x^2\cos 2\theta + 51x^3\cos 3\theta +$ &c. . . .

56. $1 + 11x\sin\alpha + 89x^2\sin 2\alpha + 659x^3\sin 3\alpha\ldots$

CHAPTER VIII.

IMAGINARY ANGLES.

SECTION I.—CIRCULAR FUNCTIONS OF IMAGINARY ANGLES.

186. *If in the Newtonian expansions*

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \&c., \quad \cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \&c.,$$

we substitute for x the complex magnitude

$$\rho(\cos\theta + i\sin\theta),$$

the results are convergent.

DEM.—We have, by De Moivre's theorem, after making the substitution,

$$\begin{aligned}\sin\rho(\cos\theta + i\sin\theta) &= \rho(\cos\theta + i\sin\theta) - \rho^3\frac{(\cos 3\theta + i\sin 3\theta)}{\underline{|3}} \\ &\quad + \rho^5\frac{(\cos 5\theta + i\sin 5\theta)}{\underline{|5}} - \&c., \qquad (660)\end{aligned}$$

$$\begin{aligned}&\cos\rho(\cos\theta + i\sin\theta) \\ &= 1 - \rho^2\frac{(\cos 2\theta + i\sin 2\theta)}{\underline{|2}} + \rho^4\frac{(\cos 4\theta + i\sin 4\theta)}{\underline{|4}} - \&c. \qquad (661)\end{aligned}$$

And it is evident that the real and the imaginary parts on the right in each equation are convergent. Hence, &c.

DEF.—*The circular functions tan, cot, sec, cosec of the complex magnitudes $\rho(\cos\theta + i\sin\theta)$ are defined by means of its sine and cosine by the same equations as those for real magnitudes.* Thus:—

$$\tan\rho(\cos\theta + i\sin\theta) = \frac{\sin\rho(\cos\theta + i\sin\theta)}{\cos\rho(\cos\theta + i\sin\theta)}, \ \&c. \qquad (662)$$

Cor.—The circular functions of pure imaginaries are defined in the same manner. Thus:—

$$\sin \eta i = i\left(\frac{\eta}{1} + \frac{\eta^3}{\lfloor 3} + \frac{\eta^5}{\lfloor 5} + \&\text{c.}\right), \tag{663}$$

$$\cos \eta i = 1 + \frac{\eta^2}{\lfloor 2} + \frac{\eta^4}{\lfloor 4} + \frac{\eta^6}{\lfloor 6} + \&\text{c., and so on.} \tag{664}$$

187. *The equation* $e^{ix} = \cos x + i \sin x$ *holds true, when* x *denotes a complex magnitude.*

Dem.—Put $x = \cos\theta + i\sin\theta$. Then we have

$$e^{ix} = e^{i(\cos\theta + i\sin\theta)} = 1 + i(\cos\theta + i\sin\theta)$$

$$- \frac{\cos 2\theta + i\sin 2\theta}{\lfloor 2} - \frac{i(\cos 3\theta + i\sin 3\theta)}{\lfloor 3} + \&\text{c.}$$

$$= 1 - \frac{\cos 2\theta + i\sin 2\theta}{\lfloor 2} + \frac{\cos 4\theta + i\sin 4\theta}{\lfloor 4} - \&\text{c.}$$

$$+ i\left\{(\cos\theta + i\sin\theta) - \frac{\cos 3\theta + i\sin 3\theta}{\lfloor 3} + \&\text{c.}\right\}$$

$$= \cos(\cos\theta + i\sin\theta) + i\sin(\cos\theta + i\sin\theta)$$

equations (660), (661)

$$= \cos x + i\sin x.$$

188. We shall now prove the fundamental properties of these functions.

We have (§ 161),

$$e^{i\theta} \cdot e^{i\theta'} = e^{i(\theta+\theta')}, \quad \text{and} \quad e^{-i\theta} \cdot e^{-i\theta'} = e^{-i(\theta+\theta')};$$

but, whether θ be simple or complex,

$$e^{i\theta} = \cos\theta + i\sin\theta. \quad (\S\ 187.)$$

Hence

$$(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') = \cos(\theta+\theta') + i\sin(\theta+\theta'),$$

and

$$(\cos\theta - i\sin\theta)(\cos\theta' - i\sin\theta') = \cos(\theta+\theta') - i\sin(\theta+\theta').$$

Therefore, multiplying, and then adding and subtracting, we get

$$\cos(\theta + \theta') = \cos\theta \,.\, \cos\theta' - \sin\theta \sin\theta', \tag{665}$$

$$\sin(\theta + \theta') = \sin\theta \,.\, \cos\theta' + \cos\theta \sin\theta'. \tag{666}$$

And these equations are valid, whether the quantities θ, θ' be simple or complex. *Hence all the results which have been deduced in Chapter II., from the formulae for the addition of arcs, hold without modification when complex magnitudes are substituted for simple quantities.*

189. In the identities

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

change θ into $i\theta$, and we get

$$\cos i\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta}), \quad \sin i\theta = \frac{i}{2}(e^{\theta} - e^{-\theta})$$

(values which may also be inferred from equations (663), (664)). Hence, substituting in the three equations

$$e^{\theta + i\theta'} = e^{\theta} \,.\, e^{i\theta'},$$

$$\cos(\theta + i\theta') = \cos\theta \,.\, \cos i\theta' - \sin\theta \,.\, \sin i\theta',$$

$$\sin(\theta + i\theta') = \sin\theta \,.\, \cos i\theta' + \cos\theta \,.\, \sin i\theta',$$

we get

$$e^{\theta + i\theta'} = e^{\theta}(\cos\theta' + i\sin\theta'), \tag{667}$$

$$\cos(\theta + i\theta') = \tfrac{1}{2}(e^{\theta'} + e^{-\theta'})\cos\theta - \frac{i}{2}(e^{\theta'} - e^{-\theta'})\sin\theta, \tag{668}$$

$$\sin(\theta + i\theta') = \tfrac{1}{2}(e^{\theta'} + e^{-\theta'})\sin\theta + \frac{i}{2}(e^{\theta'} - e^{-\theta'})\cos\theta. \tag{669}$$

Again we have, Def., § 186,

$$\tan(\theta + i\theta') = \frac{\sin(\theta + i\theta')}{\cos(\theta + i\theta')} = \frac{2\sin(\theta + i\theta')\cos(\theta - i\theta')}{2\cos(\theta + i\theta')\cos(\theta - i\theta')}$$

$$= \frac{\sin 2\theta + \sin 2i\theta'}{\cos 2\theta + \cos 2i\theta'}, \tag{670}$$

$$= \frac{2\sin 2\theta + i(e^{2\theta'} - e^{-2\theta'})}{2\cos 2\theta + e^{2\theta'} + e^{-2\theta'}}. \tag{671}$$

Cor.—$\tan i\theta = \sin i\theta/\cos i\theta = i\,(e^{\theta} - e^{-\theta})/(e^{\theta} + e^{-\theta})$. (672)

The formulae (667)-(672) are true, whether the quantities θ, θ' be simple or complex: their principal use is to reduce to ordinary forms such expressions as e^x, $\cos x$, $\sin x$, $\tan x$, when x denotes a complex magnitude.

190. From the equations (663), (669) it follows at once that the equation $\sin x = 0$ cannot be satisfied by imaginary values of x; and since 0, $\pm\ \pi$, $\pm\ 2\pi$, &c., are the real values that satisfy it, we have, by the theory of equations,

$$\sin x = Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right), \ldots \text{ to inf.},$$

where A cannot contain x; but

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \text{\&c.} \quad (\S\ 158);$$

therefore, dividing by x, we have

$$(1 - \frac{x^2}{\underline{|3}} + \frac{x^4}{\underline{|5}} - \text{\&c.}) = A\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right); \ldots$$

or, supposing the second side multiplied out,

$$(1 - \frac{x^2}{\underline{|3}} + \frac{x^4}{\underline{|5}} - \text{\&c.}) = A\,(1 - c_1x^2 + c_2x^4 - \text{\&c.})$$

Hence, comparing coefficients, we have $A = 1$;

$$\therefore\quad \sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right), \text{ \&c. to inf.}$$

This demonstration affords an instance of the utility of considering the circular functions of imaginary angles. For, as usually presented, the demonstration has the serious defect of not proving that A cannot be a function of x: besides this, it has another fault, viz., the method of ascertaining the value of A, which is as follows:—"We have

$$\frac{\sin x}{x} = A\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right). \ldots$$

Hence the limit of $\frac{\sin x}{x}$ is equal to the limit of

$$A\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\ldots,$$

when x approaches zero; but this limit is A; hence $A = 1$." But it is evident that this is imperfect, as it cannot be asserted without proof that the limit of the second side is A, since the number of factors is infinite.

As another example, *let it be required to decompose*

$$e^z - 2\cos a + e^{-z}$$

into an infinite number of factors.

It is evident that $e^z - 2\cos a + e^{-z}$ vanishes, when $z = i(2k\pi \pm a)$, and does not vanish for any real value. Hence

$$e^z - 2\cos a + e^{-z} = A\left(1+\frac{z^2}{a^2}\right)\left(1+\frac{z^2}{(2\pi \pm a)^2}\right)\left(1+\frac{z^2}{(4\pi \pm a)^2}\right)\ldots,$$

where A is a constant. This may be written

$$2(1-\cos a) + 2\left(\frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} + \&c.\right) = A(1 + c_1 z^2 + c_2 z^4 + \&c.)$$

Hence, comparing coefficients, we have

$$A = 4\sin^2\frac{a}{2};$$

therefore

$$e^z - 2\cos a + e^{-z} = 4\sin^2\frac{a}{2}\left(1+\frac{z^2}{a^2}\right)\left(1+\frac{z^2}{(2\pi \pm a)^2}\right)\left(1+\frac{z^2}{(4\pi \pm a)^2}\right)\ldots \quad (673)$$

Exercises.—XXXIX.

1. Given $\log(m+ni) = a + bi$, prove $2a = \log(m^2+n^2)$, $\tan b = \frac{n}{m}$. (674)
2. Reduce $\log(a+bi)^{(m+ni)}$ to the form $\alpha + \beta i$.
3. „ $(a+bi)^{(m+ni)}$ to the form $\alpha + \beta i$.
4. Prove $\{\sin(\alpha+\theta) - e^{\alpha i}\sin\theta\}^n = \sin^n\alpha \,.\, e^{-n\theta i}$. (675)
5. „ $\sin(\alpha + n\theta) - e^{\alpha i}\sin n\theta = e^{-in\theta} \,.\, \sin\alpha$. (676)

6. Prove $4\sin(\theta + i\theta')\cos(\theta - i\theta') = 2\sin 2\theta + i(e^{2\theta'} - e^{-2\theta'})$. (677)

7. ,, $4\cos(\theta + i\theta')\cos(\theta - i\theta') = 2\cos 2\theta + (e^{2\theta'} + e^{-2\theta'})$. (678)

8. If $\cos(\theta + i\theta) = \cos\alpha + i\sin\alpha$, prove $\sin^4\theta = \sin^2\alpha$. (679)

9. ,, $\tan(\theta + i\theta) = \cos 2\alpha + i\sin 2\alpha$, prove $e^{2\theta} = \pm\tan\left(\alpha + \frac{\pi}{4}\right)$. (680)

10. Prove $\sec(\theta + i\theta') = \dfrac{2e^{\theta'}(\cos\theta + i\sin\theta) + 2e^{-\theta'}(\cos\theta - i\sin\theta)}{2\cos 2\theta + e^{2\theta'} + e^{-2\theta'}}$. (681)

11. ,, $\operatorname{cosec}(\theta + i\theta') = \dfrac{2e^{\theta'}(\cos\theta + i\sin\theta) + 2e^{-\theta'}(\cos\theta - i\sin\theta)}{2\sin 2\theta + i(e^{2\theta'} - e^{-2\theta'})}$. (682)

12. ,, $\log\left(\dfrac{a+ib}{a-ib}\right) = 2i \,.\, \tan^{-1}\left(\dfrac{b}{a}\right)$. (683)

13. ,, $\log\left\{\dfrac{\sin(\theta + i\theta')}{\sin(\theta - i\theta')}\right\} = 2i\tan^{-1}\left\{\dfrac{e^{\theta'} - e^{-\theta'}}{e^{\theta'} + e^{-\theta'}} \,.\, \cot\theta\right\}$. (684)

14. ,, $\log\left\{\dfrac{\cos(\theta - i\theta')}{\cos(\theta + i\theta')}\right\} = 2i\tan^{-1}\left\{\dfrac{e^{2\theta'} - 1}{e^{2\theta'} + 1} \,.\, \tan\theta\right\}$. (685)

15. ,, $4\sin(\theta + i\theta')\sin(\theta - i\theta') = e^{2\theta'} + e^{-2\theta'} - 2\cos 2\theta$. (686)

16. ,, $4\cos(\theta + i\theta')\sin(\theta - i\theta') = 2\sin 2\theta - i(e^{2\theta'} - e^{-2\theta'})$. (687)

17. Given $\cos(\theta + i\theta') = a + bi$, prove that

$$\sin^4\theta = (1 - a^2 - b^2)\sin^2\theta + b^2,$$

and
$$\theta' = \log\left(\frac{a}{\cos\theta} - \frac{b}{\sin\theta}\right). \tag{688}$$

18. Express $a^{\sqrt[n]{i}}$ in the form of a complex magnitude.

$$\sqrt[n]{i} = 2n^{th} \text{ root of } (-1) = \cos\frac{(2r+1)\pi}{2n} + i\sin\frac{(2r+1)\pi}{2n},$$

and putting $a = e^k$, we get

$$e^{k\cos(2r+1)\pi/2n} \,.\, e^{ik\sin(2r+1)\pi/2n}$$

$$= a^{\cos\phi}\{\cos(k\sin\phi) + i\sin(k\sin\phi)\}, \tag{689}$$

where
$$\phi = \frac{(2r+1)\pi}{2n}, \quad \text{and} \quad k = \log_e a.$$

19. Prove

$$\frac{1}{(2\pi-\alpha)^2}+\frac{1}{(2\pi+\alpha)^2}+\frac{1}{(4\pi-\alpha)^2}+\frac{1}{(4\pi+\alpha)^2}+\frac{1}{(6\pi-\alpha)^2}+\frac{1}{(6\pi+\alpha)^2}, \text{ \&c.},$$

to inf. $$=\frac{1}{4\sin^2\alpha/2}-\frac{1}{\alpha^2}. \qquad (690)$$

20. Prove

$$\frac{1}{(\pi-\alpha)^2}+\frac{1}{(\pi+\alpha)^2}+\frac{1}{(2\pi-\alpha)^2}+\frac{1}{(2\pi+\alpha)^2}+\frac{1}{(3\pi-\alpha)^2}+\frac{1}{(3\pi+\alpha)^2}, \text{ \&c.},$$

to inf. $$=\frac{1}{\sin^2\alpha}-\frac{1}{\alpha^2}. \qquad (691)$$

21. Prove

$$\frac{1}{(\pi-\alpha)^2}+\frac{1}{(\pi+\alpha)^2}+\frac{1}{(3\pi-\alpha)^2}+\frac{1}{(3\pi+\alpha)^2}+\frac{1}{(5\pi-\alpha)^2}+\frac{1}{(5\pi+\alpha)^2}, \text{ \&c.},$$

to inf. $$=\frac{1}{4\cos^2\alpha/2}. \qquad (692)$$

22. Prove $$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}\ldots \text{ to inf.}=\frac{\pi^3}{2^5}.$$

23. ,, $$\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}\ldots \text{ to inf.}=\frac{5\pi^5}{3\,.\,2^9}.$$

Section II.—Hyperbolic Sines and Cosines.

191. Def.—*If in the exponential expressions*

$$\frac{e^{i\theta}-e^{-i\theta}}{2i}, \quad \frac{e^{i\theta}+e^{-i\theta}}{2},$$

for the circular functions of the sine and the cosine of an angle θ, we omit the i, the results are called, respectively, the hyperbolic sine and cosine of θ.

Hyperbolic functions are so called, because they have geometrical relations with the equilateral hyperbola analogous to those between the circular functions and the circle (see § 148, *Cor.* 6); also the author's *Analytical Geometry*, p. 225.

"Le P. V. de Riccati, S.J. (1707, 1775), a formé, au moyen du nombre e, certaines expressions dites *fonctions hyperboliques*, qui jouissent de propriétés semblables à celles *des fonctions circulares* sinus, cosinus, tangente, &c., mais plus faciles à établir. Plusieurs géomètres du XVIII[e] et du XIX[e] siecle (entre autres Moivre, De Foncenex, Lambert, Euler, Gudermann,

Grassmann, Lamé, Gronau, Hoüel, Laisant, Günther), que ont employé consciemment ou inconsciemment les fonctions hyperboliques ont fait ressortir l'utilité des notations de RICCATI."—(MANSION.)

192. Notation.—Hyperbolic functions are denoted as follows, viz.:—

sine hyperbolic θ,	Sh θ,	cosine hyperbolic θ,	Ch θ;
tangent hyp θ,	Th θ,	cotangent hyp θ,	Coth θ;
secant hyp θ,	Sech θ,	cosecant hyp θ,	Cosech θ.

193. From the definitions we have, evidently,

$$\text{Sh } \theta = \frac{e^{\theta} - e^{-\theta}}{2}, \tag{693}$$

$$\text{Ch } \theta = \frac{e^{\theta} + e^{-\theta}}{2}, \tag{694}$$

$$\text{Th } \theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} = \frac{e^{2\theta} - 1}{e^{2\theta} + 1}, \tag{695}$$

$$\text{Coth } \theta = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}} = \frac{e^{2\theta} + 1}{e^{2\theta} - 1}, \tag{696}$$

$$\text{Sech } \theta = \frac{1}{\text{Ch } \theta} = \frac{2}{e^{\theta} + e^{-\theta}} = \frac{2e^{\theta}}{e^{2\theta} + 1}, \tag{697}$$

$$\text{Cosech } \theta = \frac{1}{\text{Sh } \theta} = \frac{2}{e^{\theta} - e^{-\theta}} = \frac{2e^{\theta}}{e^{2\theta} - 1}. \tag{698}$$

194. If we expand the right side of (694), we see that Ch θ contains only even powers of θ. Hence Ch θ remains unaltered when θ changes sign. Therefore we have

$$\text{Ch } (-\theta) = \text{Ch } \theta; \quad \text{Sech } (-\theta) = \text{Sech } \theta. \tag{699}$$

Similarly,

$$\text{Sh } (-\theta) = -\text{ Sh } \theta, \quad \text{Cosech } (-\theta) = -\text{ cosec } \theta, \tag{700}$$

and

$$\text{Th } (-\theta) = -\text{ Th } \theta, \quad \text{Coth } (-\theta) = -\text{ Coth } \theta. \tag{701}$$

Hence, when the sign of the argument is changed, the hyperbolic functions follow the same rules of change or permanence as the corresponding circular functions.

195. From the exponential values we get the following formulae for the sums and differences (compare §§ 39, 40, 41):—

$$\mathrm{Sh}\,(\alpha+\beta) = \mathrm{Sh}\,\alpha\,.\,\mathrm{Ch}\,\beta + \mathrm{Ch}\,\alpha\,.\,\mathrm{Sh}\,\beta, \qquad (702)$$

$$\mathrm{Sh}\,(\alpha-\beta) = \mathrm{Sh}\,\alpha\,.\,\mathrm{Ch}\,\beta - \mathrm{Ch}\,\alpha\,.\,\mathrm{Sh}\,\beta, \qquad (703)$$

$$\mathrm{Ch}\,(\alpha+\beta) = \mathrm{Ch}\,\alpha\,.\,\mathrm{Ch}\,\beta + \mathrm{Sh}\,\alpha\,.\,\mathrm{Sh}\,\beta, \qquad (704)$$

$$\mathrm{Ch}\,(\alpha-\beta) = \mathrm{Ch}\,\alpha\,.\,\mathrm{Ch}\,\beta - \mathrm{Sh}\,\alpha\,.\,\mathrm{Sh}\,\beta. \qquad (705)$$

196. From the formulae (702)–(705), which may be called the four fundamental formulae of hyperbolic functions, we get, as in the analogous case of circular functions,

$$\mathrm{Sh}\,(\alpha+\beta) + \mathrm{Sh}\,(\alpha-\beta) = 2\,\mathrm{Sh}\,\alpha\,.\,\mathrm{Ch}\,\beta, \qquad (706)$$

$$\mathrm{Sh}\,(\alpha+\beta) - \mathrm{Sh}\,(\alpha-\beta) = 2\,\mathrm{Ch}\,\alpha\,.\,\mathrm{Sh}\,\beta, \qquad (707)$$

$$\mathrm{Ch}\,(\alpha+\beta) + \mathrm{Ch}\,(\alpha-\beta) = 2\,\mathrm{Ch}\,\alpha\,.\,\mathrm{Ch}\,\beta, \qquad (708)$$

$$\mathrm{Ch}\,(\alpha+\beta) - \mathrm{Ch}\,(\alpha-\beta) = 2\,\mathrm{Sh}\,\alpha\,.\,\mathrm{Sh}\,\beta. \qquad (709)$$

197. If we put $\alpha+\beta=\alpha'$, and $\alpha-\beta=\beta'$, we get

$$\alpha = \tfrac{1}{2}(\alpha'+\beta'), \quad \beta = \tfrac{1}{2}(\alpha'-\beta').$$

Hence, making these substitutions in the formulae (706)–(709), and omitting accents, we get the following formulae for the transformation of sums and differences into products:—

$$\mathrm{Sh}\,\alpha + \mathrm{Sh}\,\beta = 2\,\mathrm{Sh}\,\tfrac{1}{2}(\alpha+\beta)\,\mathrm{Ch}\,\tfrac{1}{2}(\alpha-\beta), \qquad (710)$$

$$\mathrm{Sh}\,\alpha - \mathrm{Sh}\,\beta = 2\,\mathrm{Ch}\,\tfrac{1}{2}(\alpha+\beta)\,\mathrm{Sh}\,\tfrac{1}{2}(\alpha-\beta), \qquad (711)$$

$$\mathrm{Ch}\,\alpha + \mathrm{Ch}\,\beta = 2\,\mathrm{Ch}\,\tfrac{1}{2}(\alpha+\beta)\,\mathrm{Ch}\,\tfrac{1}{2}(\alpha-\beta), \qquad (712)$$

$$\mathrm{Ch}\,\alpha - \mathrm{Ch}\,\beta = 2\,\mathrm{Sh}\,\tfrac{1}{2}(\alpha+\beta)\,\mathrm{Sh}\,\tfrac{1}{2}(\alpha-\beta). \qquad (713)$$

198. From the formulae for two angles we may proceed as in the corresponding case of circular functions, and investigate formulae for three angles, &c. Thus:—

$$\begin{aligned}\mathrm{Sh}\,(\alpha+\beta+\gamma) &= \mathrm{Sh}\,\alpha\,.\,\mathrm{Ch}\,\beta\,.\,\mathrm{Ch}\,\gamma + \mathrm{Sh}\,\beta\,.\,\mathrm{Ch}\,\gamma\,.\,\mathrm{Ch}\,\alpha \\ &\quad + \mathrm{Sh}\,\gamma\,.\,\mathrm{Ch}\,\alpha\,.\,\mathrm{Ch}\,\beta + \mathrm{Sh}\,\alpha\,.\,\mathrm{Sh}\,\beta\;\mathrm{Sh}\,\gamma, \qquad (714)\end{aligned}$$

and

$$\text{Ch}(\alpha+\beta+\gamma) = \text{Ch}\,\alpha\,.\,\text{Ch}\,\beta\,.\,\text{Ch}\,\gamma + \text{Ch}\,\alpha\,\text{Sh}\,\beta\,\text{Sh}\,\gamma$$
$$+ \text{Ch}\,\beta\,\text{Sh}\,\gamma\,\text{Sh}\,\alpha + \text{Ch}\,\gamma\,\text{Sh}\,\alpha\,.\,\text{Sh}\,\beta. \qquad (715)$$

199. From the sines and cosines we can infer formulae for Th, Coth, &c. Thus:—

$$\text{Th}(\alpha+\beta) = \frac{\text{Th}\,\alpha + \text{Th}\,\beta}{1 + \text{Th}\,\alpha\ \text{Th}\,\beta}. \qquad (716)$$

$$\text{Th}(\alpha-\beta) = \frac{\text{Th}\,\alpha - \text{Th}\,\beta}{1 - \text{Th}\,\alpha\,.\,\text{Th}\,\beta}. \qquad (717)$$

$$\text{Th}(\alpha+\beta+\gamma) = \frac{\text{Th}\,\alpha + \text{Th}\,\beta + \text{Th}\,\gamma + \text{Th}\,\alpha\,\text{Th}\,\beta\,\text{Th}\,\gamma}{1 + \text{Th}\,\alpha\,\text{Th}\,\beta + \text{Th}\,\beta\,.\,\text{Th}\,\gamma + \text{Th}\,\gamma\,\text{Th}\,\alpha}. \qquad (718)$$

$$\text{Th}\,\alpha + \text{Th}\,\beta = \frac{\text{Sh}(\alpha+\beta)}{\text{Ch}\,\alpha\,.\,\text{Ch}\,\beta}. \qquad (719)$$

$$\text{Th}\,\alpha - \text{Th}\,\beta = \frac{\text{Sh}(\alpha-\beta)}{\text{Ch}\,\alpha\,.\,\text{Ch}\,\beta}. \qquad (720)$$

200. *Connexion between the hyperbolic functions of a real, and the circular functions of an imaginary, angle.*

If we compare § 189 and § 193, we get the following relations:—

$$\text{Ch}(\alpha) = \cos(i\alpha), \quad i\,\text{Sh}\,\alpha = \sin(i\alpha), \quad i\,\text{Th}\,\alpha = \tan(i\alpha). \quad (721)$$

Hence we have the following rule of transformation, viz.:—

In any identity of circular functions change θ into $i\theta$, and then substitute $\text{Ch}\,\theta$ for $\cos(i\theta)$, $i\,\text{Sh}\,\theta$ for $\sin i\theta$, and $i\,\text{Th}\,\theta$ for $\tan(i\theta)$. The formulae (702)–(720), already proved, may be obtained in this manner.

Cor.—$\cos(i\theta)$ and $\dfrac{\sin i\theta}{i}$ are real.

EXERCISES.—XL.

1. Prove $\sin(\alpha + i\beta) = \sin\alpha \,.\, \mathrm{Ch}\,\beta + i\cos\alpha \,.\, \mathrm{Sh}\,\beta.$ (722)

2. ,, $\cos(\alpha + i\beta) = \cos\alpha \,.\, \mathrm{Ch}\,\beta - i\sin\alpha \,.\, \mathrm{Sh}\,\beta.$ (723)

3. ,, $\sin(\alpha - i\beta) = \sin\alpha \,.\, \mathrm{Ch}\,\beta - i\cos\alpha \,.\, \mathrm{Sh}\,\beta.$ (724)

4. ,, $\cos(\alpha - i\beta) = \cos\alpha \,.\, \mathrm{Ch}\,\beta + i\sin\alpha \,.\, \mathrm{Sh}\,\beta.$ (725)

5. ,, $\mathrm{Sh}(\alpha + i\beta) = \mathrm{Sh}\,\alpha \,.\, \cos\beta + i\,\mathrm{Ch}\,\alpha \,.\, \sin\beta.$ (726)

6. ,, $\mathrm{Ch}(\alpha + i\beta) = \mathrm{Ch}\,\alpha \,.\, \cos\beta + i\,\mathrm{Sh}\,\alpha \,.\, \sin\beta.$ (727)

7. ,, $\mathrm{Sh}(\alpha - i\beta) = \mathrm{Sh}\,\alpha \,.\, \cos\beta - i\,\mathrm{Ch}\,\alpha \,.\, \sin\beta.$ (728)

8. ,, $\mathrm{Ch}(\alpha - i\beta) = \mathrm{Ch}\,\alpha \,.\, \cos\beta - i\,\mathrm{Sh}\,\alpha \,.\, \sin\beta.$ (729)

These formulae hold for all values, real or imaginary, of α and β. If α and β are both real, they give us the sine and the cosine, both circular and hyperbolic, of complex angles.

9. Prove $\mathrm{Sh}\,2\alpha = 2\,\mathrm{Sh}\,\alpha\,\mathrm{Ch}\,\alpha.$ (730)

10. ,, $\mathrm{Ch}\,2\alpha = \mathrm{Ch}^2\alpha + \mathrm{Sh}^2\alpha.$ (731)

11. ,, $\mathrm{Ch}^2\alpha - \mathrm{Sh}^2\alpha = 1.$ (732)

12. ,, $\mathrm{Sech}^2\alpha + \mathrm{Th}^2\alpha = 1.$ (733)

13. ,, $\mathrm{Cosech}^2\alpha + \mathrm{Coth}^2 = 1.$ (734)

14. ,, $\mathrm{Sh}\,\alpha\,\mathrm{Sh}(\beta - \gamma) + \mathrm{Sh}\,\beta\,\mathrm{Sh}(\gamma - \alpha) + \mathrm{Sh}\,\gamma\,\mathrm{Sh}(\alpha - \beta) = 0.$ (735)

15. ,, $1 + \mathrm{Ch}\,2\alpha = 2\,\mathrm{Ch}^2\alpha.$ (736)

16. ,, $\mathrm{Ch}\,2\alpha - 1 = 2\,\mathrm{Sh}^2\alpha.$ (737)

17. ,, $\mathrm{Ch}\,\alpha \,.\, \mathrm{Sh}(\beta - \gamma) + \mathrm{Ch}\,\beta\,\mathrm{Sh}(\gamma - \alpha) + \mathrm{Ch}\,\gamma\,\mathrm{Sh}(\alpha - \beta) = 0.$ (738)

18. ,, $\mathrm{Sh}(\alpha + \beta)\,\mathrm{Sh}(\alpha - \beta) = \mathrm{Sh}^2\alpha - \mathrm{Sh}^2\beta.$ (739)

19. ,, $\mathrm{Ch}(\alpha + \beta)\,\mathrm{Ch}(\alpha - \beta) = \mathrm{Sh}^2\alpha + \mathrm{Ch}^2\beta = \mathrm{Ch}^2\alpha + \mathrm{Sh}^2\beta.$ (740)

20. ,, $\mathrm{Th}\,(2\alpha) = \dfrac{2\,\mathrm{Th}\,\alpha}{1 + \mathrm{Th}^2\alpha}.$ (741)

21. ,, $\mathrm{Sh}(\alpha \pm 2in\pi) = \mathrm{Sh}\,\alpha.$ (742)

22. ,, $\mathrm{Ch}(\alpha \pm 2in\pi) = \mathrm{Ch}\,\alpha.$ (743)

23. Prove $\quad \text{Sh}\,3\alpha = 3\,\text{Sh}\,\alpha + 4\,\text{Sh}^3\alpha.$ (744)

24. ,, $\quad \text{Ch}\,3\alpha = 4\,\text{Ch}^3\alpha - 3\,\text{Ch}\,\alpha.$ (745)

25. ,, $\quad \text{Th}\,3\alpha = \dfrac{3\,\text{Th}\,\alpha + \text{Th}^3\alpha}{1 + 3\,\text{Th}^2\alpha}.$ (746)

26. ,, $\quad \text{Ch}\,\alpha + \text{Sh}\,\alpha = e^{\alpha}.$ (747)

27. ,, $\quad \text{Ch}\,\alpha - \text{Sh}\,\alpha = e^{-\alpha}.$ (748)

28. ,, $\quad (\text{Ch}\,\alpha + \text{Sh}\,\alpha)^n = \text{Ch}\,(n\alpha) + \text{Sh}\,(n\alpha).$ (749)

29. ,, $\quad (\text{Ch}\,\alpha - \text{Sh}\,\alpha)^n = \text{Ch}\,(n\alpha) - \text{Sh}\,(n\alpha).$ (750)

30. ,, $\quad \text{Sh}\,(n\alpha) = n\,\text{Ch}^{n-1}\alpha\,\text{Sh}\,\alpha + \dfrac{\lfloor n}{\lfloor 3\,\lfloor n-3}\,\text{Ch}^{n-3}\alpha\,\text{Sh}^3\alpha,$ &c. (751)

31. ,, $\quad \text{Ch}\,(n\alpha) = \text{Ch}^n\alpha + \dfrac{\lfloor n}{\lfloor 2\,\lfloor n-2}\,\text{Ch}^{n-2}\alpha\,\text{Sh}^2\alpha +$ &c. (752)

32. ,, $\quad \alpha = \text{Sh}\,\alpha - \frac{1}{2}\,\dfrac{\text{Sh}^3\alpha}{3} + \dfrac{1\,.\,3}{2\,.\,4}\,\dfrac{\text{Sh}^5\alpha}{5} -$ &c. (753)

33. ,, $\quad \alpha = \text{Th}\,\alpha + \dfrac{\text{Th}^3\alpha}{3} + \dfrac{\text{Th}^5\alpha}{5} +$ &c. (754)

34. ,, $\quad 2\,\text{Ch}\,(n\alpha) = (2\,\text{Ch}\,\alpha)^n - n\,(2\,\text{Ch}\,\alpha)^{n-2}$

$\qquad + \dfrac{n\,.\,n-1}{\lfloor 2}\,(2\,\text{Ch}\,\alpha)^{n-4} -$ &c. (755)

35. ,, $\quad 2^{n-1}\,\text{Ch}^n\alpha = \text{Ch}\,(n\alpha) + n\,\text{Ch}\,(n-2)\,\alpha$

$\qquad + \dfrac{n\,.\,n-1}{\lfloor 2}\,\text{Ch}\,(n-4)\,\alpha,$ &c. (756)

201. Hyperbolic Functions expressed by means of Circular Functions.

If in the formula $\text{Sech}^2\theta + \text{Th}^2\theta = 1$, equation (733), we put $\text{Sech}\,\theta = \cos\tau$, we get $\text{Th}\,\theta = \sin\tau$. Hence $\text{Ch}\,\theta = \sec\tau$, $\text{Sh}\,\theta = \tan\tau$, $\text{Cosech}\,\theta = \cot\tau$, $\text{Coth}\,\theta = \text{cosec}\,\tau$. (757)

The angle τ is an important one in the theory of hyperbolic functions. It has been differently named by mathematicians. Hoüel has called it the *Hyperbolic Amplitude of* θ. Gudermann, the *Longitude*. Lambert and others, the *Transcendent angle*. We prefer Hoüel's nomenclature. It is written thus:—

$$\tau = \text{Amh.}\,\theta.$$

Exercises.—XLI.

1\. Prove $$\text{Th}\,\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\tau. \qquad \text{(Laisant.)}$$

From the formulae $\text{Ch}\,2\theta = \text{Ch}^2\theta + \text{Sh}^2\theta,\quad 1 = \text{Ch}^2\theta - \text{Sh}^2\theta,$

we get $$\text{Th}^2\theta = \frac{\text{Ch}\,2\theta - 1}{\text{Ch}\,2\theta + 1}.$$

Hence, changing θ into $\tfrac{1}{2}\theta$, we get

$$\text{Th}^2\tfrac{1}{2}\theta = \frac{\text{Ch}\,\theta - 1}{\text{Ch}\,\theta + 1} = \frac{\sec\tau - 1}{\sec\tau + 1} = \frac{1 - \cos\tau}{1 + \cos\tau}.$$

Therefore $$\text{Th}\,\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\tau. \tag{758}$$

2\. Prove
$$\sin(\tau + i\theta) = \tan\tau + i\sin\tau = \text{Sh}\,\theta + i\,\text{Th}\,\theta = \text{Sh}\,\theta + i\sin\tau. \tag{759}$$

3\. Prove
$$\sin(\tau - i\theta) = \tan\tau - i\sin\tau = \text{Sh}\,\theta - i\,\text{Th}\,\theta = \text{Sh}\,\theta - i\sin\tau. \tag{760}$$

4\. Prove
$$\cos(\tau + i\theta) = 1 - i\sin\tau\tan\tau = 1 - i\,\text{Sh}\,\theta\,\text{Th}\,\theta = 1 - i\sin\tau\,\text{Sh}\,\theta. \tag{761}$$

5\. Prove
$$\cos(\tau - i\theta) = 1 + i\sin\tau\tan\tau = 1 + i\,\text{Sh}\,\theta\,.\,\text{Th}\,\theta = 1 + i\sin\tau\,\text{Sh}\,\theta. \tag{762}$$

6\. Prove
$$\text{Sh}\,(\theta + i\tau) = \text{Th}\,\theta + i\,\text{Sh}\,\theta = \sin\tau + i\tan\tau = \sin\tau + i\,\text{Sh}\,\theta. \tag{763}$$

7\. Prove
$$\text{Sh}\,(\theta - i\tau) = \text{Th}\,\theta - i\,\text{Sh}\,\theta = \sin\tau - i\tan\tau = \sin\tau - i\,\text{Sh}\,\theta. \tag{764}$$

8\. Prove $$\text{Ch}\,(\theta + i\tau) = 1 + i\,\text{Sh}\,\theta\,\text{Th}\,\theta = \cos(\tau - i\theta). \tag{765}$$

9\. ,, $$\text{Ch}\,(\theta - i\tau) = 1 - i\,\text{Sh}\,\theta\,\text{Th}\,\theta = \cos(\tau + i\theta). \tag{766}$$

10\. ,, $$\sin(\tau + i\theta) + \sin(\tau - i\theta) = 2\tan\tau. \tag{767}$$

11\. ,, $$\sin(\tau + i\theta) - \sin(\tau - i\theta) = 2i\sin\tau. \tag{768}$$

12\. ,, $$\cos(\tau + i\theta) + \cos(\tau - i\theta) = 2. \tag{769}$$

13\. ,, $$\cos(\tau + i\theta) - \cos(\tau - i\theta) = -2i\sin\tau\tan\tau. \tag{770}$$

14. Prove $\sin(\tau + i\theta)\sin(\tau - i\theta) = \tan^2\tau + \sin^2\tau.$

15. ,, $\cos(\tau + i\theta)\cos(\tau - i\theta) = 1 + \sin^2\tau \, . \tan^2\tau.$

16. ,, $\text{Sh}(\theta + i\tau) + \text{Sh}(\theta - i\tau) = 2\,\text{Th}\,\theta.$

17. ,, $\text{Sh}(\theta + i\tau) - \text{Sh}(\theta - i\tau) = 2i\,\text{Sh}\,\theta.$

18. ,, $\text{Ch}(\theta + i\tau) + \text{Ch}(\theta - i\tau) = 2.$

19. ,, $\text{Ch}(\theta + i\tau) - \text{Ch}(\theta - i\tau) = 2\,\text{Sh}\,\theta \, . \, \text{Th}\,\theta.$

20. ,, $$\text{Sh}(\theta + i\tau) \, . \, \text{Sh}(\theta - i\tau) = \text{Th}^2\theta + \text{Sh}^2\theta$$
$$= \sin^2\tau + \tan^2\tau = \sin^2\tau + \text{Sh}^2\theta.$$

21. ,, $$\text{Ch}(\theta + i\tau)\,\text{Ch}(\theta - i\tau) = 1 + \text{Sh}^2\theta\,\text{Th}^2\theta$$
$$= 1 + \sin^2\tau \, . \tan^2\tau = 1 + \text{Sh}^2\theta \, . \sin^2\tau.$$

202. Tables of Hyperbolic Functions.

Since $e^\theta = \text{Ch}\,\theta + \text{Sh}\,\theta = \sec\tau + \tan\tau = \tan\left(\frac{\tau}{2} + \frac{\pi}{4}\right),$

we have
$$\theta = \log_e \tan\left(\frac{\tau}{2} + \frac{\pi}{4}\right). \tag{771}$$

From this formula it is evident that we can transform a Table of circular functions into one of hyperbolic functions, and that for that purpose it is sufficient to add a column containing the values of θ corresponding to the different values of τ from 0° to 90°. This will be fully understood from the following specimen taken from Hoüel's Tables (Paris, Gauthier-Villais, 1884) :—

FORM OF TABLE OF HYPERBOLIC AND CIRCULAR FUNCTIONS.

θ	—	Th θ	Coth θ	Sh θ	Cosech θ	Ch θ	Sech θ	—	×
—	τ	sin τ	cosec τ	tan τ	cot τ	sec τ	cos τ	×	—
0·0000	0°	0·0000	∞	0·0000	∞	1·0000	1·0000	90°	∞
0·0175	1°	0·0175	57·2987	0·0175	52·2900	1·0002	0·9998	89°	4·7413
0·0349	2°	0·0349	28·6537	0·0349	28·6363	1·0006	0·9994	88°	4·0481
0·0524	3°	0·0523	19·1073	0·0524	19·0811	1·0014	0·9986	87°	3·6425
0·0699	4°	0·0698	14·3356	0·0699	14·3007	1·0024	0·9976	86°	3·3548
0·0874	5°	0·0872	11·4737	0·0875	11·4301	1·0038	0·9962	85°	3·1313
—	×	cos τ	sec τ	cot τ	tan τ	cosec τ	sin τ	τ	—
×	—	Sech θ	Ch θ	Cosech θ	Sh θ	Coth θ	Th θ	—	θ

203. If in the formula $\text{Ch}^2\theta - \text{Sh}^2\theta = 1$, we put $\text{Sh}\,\theta = 1$, we get

$$\text{Ch}\,\theta = \sqrt{2};$$

and since $e^\theta = \text{Ch}\,\theta + \text{Sh}\,\theta$, $e^\theta = 1 + \sqrt{2}$;

therefore
$$\theta = \log_e(1 + \sqrt{2}).$$

Hence, denoting, as Laisant does, the value of θ, whose hyperbolic sine is equal to unity by Π, we have

$$\Pi = \log_e(1 + \sqrt{2}). \tag{772}$$

Cor. 1.— $\text{Ch}\,\Pi = \sqrt{2}$, $\text{Th}\,\Pi = \frac{1}{\sqrt{2}} = \text{Sech}\,\Pi.$ (773)

Cor. 2.—If in equation (753) we put $\text{Sh}\,\alpha = 1$, we have $\alpha = \Pi$. Hence, from (772),

$$\log_e(1 + \sqrt{2}) = 1 - \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1.3}{2.4} - \frac{1}{7}\cdot\frac{1.3.5}{2.4.6}, \text{ \&c.} \tag{774}$$

Cor. 3.— $\text{Amh}\,\Pi = \frac{\pi}{4}.$ (775)

Cor. 4.— $\text{Th}\,\frac{\Pi}{2} = e^{-\Pi}.$ (776)

Dem.— $$\text{Th}\,\frac{\Pi}{2} = \tan\frac{\pi}{8} = \sqrt{2} - 1 = \frac{1}{1+\sqrt{2}} = e^{-\Pi}.$$

Cor. 5—In equation (754), put $\alpha = \Pi$.

We get $$\Pi\sqrt{2} = 1 + \frac{1}{3.2} + \frac{1}{5.2^2} + \frac{1}{7.2^3} + \frac{1}{9.2^4}\ldots$$

Cor. 6.—$$\text{Ch}(n\Pi)/\text{Ch}^n\Pi = 1 + \frac{n.n-1}{2\lfloor 2} + \frac{n\,n-1.n-2.n-3}{4\lfloor 4} + \text{\&c.}$$

This follows from equation (752).

Exercises XLII.

1. Prove

$$\operatorname{Ch} n\theta = 1 + \frac{n^2}{\lfloor 2}\operatorname{Sh}^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\operatorname{Sh}^4\theta + \frac{n^2(n^2-2^2)(n^2-4^2)}{\lfloor 6}\operatorname{Sh}^6\theta, \text{ \&c.} \quad (777)$$

2. Find the sum of the series

$$1 + \frac{n^2}{\lfloor 2} + \frac{n^2(n^2-2^2)}{\lfloor 4} + \text{\&c.}$$

3. Prove

$$\operatorname{Sh} n\theta = n\operatorname{Sh}\theta + \frac{n(n^2-1^2)}{\lfloor 3}\operatorname{Sh}^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\operatorname{Sh}^5\theta, \text{ \&c.} \quad (778)$$

4. Find the sum of the series

$$1 + \frac{n^2-1^2}{\lfloor 3} + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 5}, \text{ \&c.}$$

5. Prove

$$\operatorname{Sh} n\theta = \operatorname{Ch}^n\theta\left\{n\operatorname{Th}\theta + \frac{\lfloor n}{\lfloor 3\,\lfloor n-3}\operatorname{Th}^3\theta + \frac{\lfloor n}{\lfloor 5\,\lfloor n-5}\operatorname{Th}^5\theta + \text{\&c.}\right\}. \quad (779)$$

6. Prove

$$\operatorname{Ch} n\theta = \operatorname{Ch}^n\theta\left\{1 + \frac{\lfloor n}{\lfloor 2\,\lfloor n-2}\operatorname{Th}^2\theta + \frac{\lfloor n}{\lfloor 4\,\lfloor n-4}\operatorname{Th}^4\theta + \text{\&c.}\right\}. \quad (780)$$

7. If we put $\tan\frac{1}{2}\tau = t$, we have $\operatorname{Th}\dfrac{\theta}{2} = t$. Hence, prove

$$\frac{\theta+\tau}{4} = t + \frac{t^5}{5} + \frac{t^9}{9} + \frac{t^{13}}{13} + \text{\&c.} \quad (781)$$

8. And

$$\frac{\theta-t}{4} = \frac{t^3}{3} + \frac{t^7}{7} + \frac{t^{11}}{11} + \frac{t^{15}}{15}, \text{ \&c.} \quad (782)$$

9. Prove

$$\frac{4\Pi+\pi}{16} = e^{\Pi} + \frac{1}{5}e^{5\Pi} + \frac{1}{9}e^{-9\Pi} + \text{\&c.} \quad (783)$$

10. ,,

$$\frac{4\Pi-\pi}{16} = \frac{1}{3}e^{-3\Pi} + \frac{1}{7}e^{-7\Pi} + \frac{1}{11}e^{-11\Pi} + \text{\&c.} \quad (784)$$

11. ,,

$$\operatorname{Th}\theta/(8\theta) = \frac{1}{\pi^2+4\theta^2} + \frac{1}{9\pi^2+4\theta^2} + \frac{1}{25\pi^2+4\theta^2} + \text{\&c.} \quad (785)$$

12. ,,

$$1 = \Pi\cdot\left(1+\frac{\Pi^2}{\pi^2}\right)\left(1+\frac{\Pi^2}{4\pi^2}\right)\left(1+\frac{\Pi^2}{9\pi^2}\right)\cdots \quad (786)$$

13. If $\sin\tau = \Theta$, prove

$$\theta + \tau = 2\Theta + \frac{\underline{|2} + 1^2}{\underline{|8}}\,\Theta^3 + \frac{\underline{|4} + (1\,.\,3)^2}{\underline{|5}}\,\Theta^5 + \&c. \tag{787}$$

14. And

$$\theta - \tau = \frac{\underline{|2} - 1^2}{\underline{|3}}\,\Theta^3 + \frac{\underline{|4} - (1\,.\,3)^2}{\underline{|5}}\,\Theta^5 + \frac{\underline{|6} - (1\,.\,3\,.\,5)^2}{\underline{|7}}\,\Theta^7, \&c. \tag{788}$$

15. Prove $$\frac{4\Pi + \pi}{2\sqrt{2}} = 2 + \frac{1}{2}\,\frac{\underline{|2} + 1^2}{\underline{|3}} + \frac{1}{2^2}\,\frac{\underline{|4} + (1\,.\,3)^2}{\underline{|5}} + \&c. \tag{789}$$

16. Prove

$$\frac{4\Pi - \pi}{\sqrt{2}} = \frac{\underline{|2} - 1^2}{\underline{|3}} + \frac{1}{2}\,\frac{\underline{|4} - (1\,.\,3)^2}{\underline{|5}} + \frac{1}{2^2}\,\frac{\underline{|6} - (1\,.\,3\,.\,5)^2}{\underline{|7}} + \&c. \tag{790}$$

17. If $\mathrm{Sh}\,\theta = \phi$, prove

$$\theta + \tau = 2\phi - \frac{\underline{|2} + 1^2}{\underline{|3}}\,\phi^3 + \frac{\underline{|4} + (1\,.\,3)^2}{\underline{|5}}\,\phi^5, \&c. \tag{791}$$

18. And

$$\theta - \tau = \frac{\underline{|2} - 1^2}{\underline{|3}}\,\phi^3 - \frac{\underline{|4} - (1\,.\,3)^2}{\underline{|5}} + \frac{\underline{|6} - (1\,.\,3\,.\,5)^2}{\underline{|7}}\,\phi^7 - \&c. \tag{792}$$

19. Prove

$$4\Pi + \pi = 4\left\{2 - \frac{\underline{|2} + 1^2}{\underline{|3}} + \frac{\underline{|4} + (1\,.\,3)^2}{\underline{|5}} - \frac{\underline{|6} + (1\,.\,3\,.\,5)^2}{\underline{|7}}, \&c.\right\} \tag{793}$$

20. Prove

$$4\Pi - \pi = 4\left\{\frac{\underline{|2} - 1^2}{\underline{|3}} - \frac{\underline{|4} - (1\,.\,3)^2}{\underline{|5}} + \frac{\underline{|6} - (1\,.\,3\,.\,5)^2}{\underline{|7}} - \&c.\right\} \tag{794}$$

21. Prove

$$\theta^2 = \mathrm{Sh}^2\theta - \frac{1}{2}\cdot\frac{2^2}{\underline{|3}}\,\mathrm{Sh}^4\theta + \frac{1}{3}\,\frac{(2\,.\,4)^2}{\underline{|5}}\,\mathrm{Sh}^6\theta - \&c. \tag{795}$$

[Make use of § 200.]

22. Prove

$$\Pi^2 = 1 - \frac{1}{2}\cdot\frac{2^2}{\underline{|3}} + \frac{2^4}{3}\cdot\frac{(\underline{|2})^2}{\underline{|5}} - \frac{2^6}{4}\,\frac{(\underline{|3})^2}{\underline{|7}} + \frac{2^8}{5}\cdot\frac{(\underline{|4})^2}{\underline{|9}}, \&c. \tag{796}$$

23. Prove $$\mathrm{Sh}\,\theta = \theta\left(1 + \frac{\theta^2}{\pi^2}\right)\left(1 + \frac{\theta^2}{4\pi^2}\right)\left(1 + \frac{\theta^2}{9\pi^2}\right)\ldots \tag{797}$$

24. ,, $$\mathrm{Ch}\,\theta = \left(1 + \frac{4}{\pi^2}\right)\left(1 + \frac{4\theta^2}{9\pi^2}\right)\left(1 + \frac{4\theta^2}{25\pi^2}\right)\ldots \tag{798}$$

25. Prove $\sqrt{2} = \left(1+\frac{4\Pi^2}{\pi^2}\right)\left(1+\frac{4\Pi^2}{9\pi^2}\right)\left(1+\frac{4\Pi^2}{25\pi^2}\right)\dots$ (799)

26. ,, $\left(e+\frac{1}{e}\right) = 2\left(1+\frac{4}{\pi^2}\right)\left(1+\frac{4}{9\pi^2}\right)\left(1+\frac{4}{25\pi^2}\right)\dots$ (800)

27. ,, $\frac{1}{2\pi}(e^{\pi}-e^{-\pi}) = \frac{1^2+1}{1^2}\times\frac{2^2+1}{2^2}\times\frac{3^2+1}{3^2}\dots$ (801)

28. ,, $\frac{1}{2}(e^{\pi}+e^{-\pi}) = \frac{2^2+1^2}{1^2}\cdot\frac{2^2+3^2}{3^2}\times\frac{2^2+5^2}{5^2}\dots$ (802)

29. ,, $\frac{1}{2\theta}(\operatorname{Coth}\theta-\theta^{-1}) = \frac{1}{\pi^2+\theta^2}+\frac{1}{4\pi^2+\theta^2}+\frac{1}{9\pi^2+\theta^2}.$ (803)

30. Prove

$$\frac{1}{\Pi\sqrt{2}}\left(1-\frac{1}{\Pi\sqrt{2}}\right) = \frac{1}{\pi^2+\Pi^2}+\frac{1}{4\pi^2+\Pi^2}+\frac{1}{9\pi^2+\Pi^2}+\&c. \qquad (804)$$

31. Prove

$$\operatorname{Th}\alpha = \frac{2^2(2^2-1)}{\lfloor 2}B_1\alpha - \frac{2^4(2^4-1)}{\lfloor 4}B_2\alpha^3 + \frac{2^6(2^6-1)}{\lfloor 6}B_3\alpha^5 - \&c. \qquad (805)$$

32. Prove $\frac{1}{e^2-1} = \frac{1}{\pi^2+1}+\frac{1}{4\pi^2+1}+\frac{1}{9\pi^2+1}+\&c.$ (806)

33. ,, $1/(8\Pi\sqrt{2}) = \frac{1}{\pi^2+4\Pi^2}+\frac{1}{9\pi^2+4\Pi^2}+\frac{1}{25\pi^2+4\Pi^2}+\&c.$ (807)

34. ,, $(e^2-1)/8(e^2+1) = \frac{1}{\pi^2+4}+\frac{1}{9\pi^2+4}+\frac{1}{25\pi^2+4}\dots$ (808)

35. ,, $\pi\operatorname{Th}\pi\alpha/8\alpha = \frac{1}{1+4\alpha^2}+\frac{1}{9+4\alpha^2}+\frac{1}{25+4\alpha^2}+\&c.$ (809)

36. ,, $\operatorname{Th}\alpha = \frac{\alpha}{1+}\ \frac{\alpha^2}{3+}\ \frac{\alpha^2}{5+}\ \frac{\alpha^2}{7+}$, &c. (810)

ANSWERS TO EXERCISES.

Exercises.—I. Page 6.

1. 108°. **2.** $3\pi/4$. **3.** $\pi/16$. **4.** 238793.

5 92678844. **6.** 431643, 431940. **7.** 25494329280000.

Exercises.—III. Page 18.

1. 80°, 33° **2.** $\sin(\pi/2+\theta)=\cos\theta$, $\cos(\pi/2+\theta)=-\sin\theta$, &c.

3. $\sin\{(4n+1)\pi/2-\theta\}=\cos\theta$, $\cos\{(4n+1)\pi/2-\theta\}=\sin\theta$, &c.,
$\sin\{(4n+3)\pi/2-\theta\}=-\cos\theta$, $\cos\{(4n+3)\pi/2-\theta\}=-\sin\theta$, &c.

4. $\sin\{(4n+1)\pi/2+\theta\}=\cos\theta$, $\cos\{(4n+1)\pi/2+\theta\}=-\sin\theta$, &c.,
$\sin\{(4n+3)\pi/2+\theta\}=-\cos\theta$, $\cos\{(4n+3)\pi/2+\theta\}=\sin\theta$, &c.

5. $-\sin 170°$, $\cot 180°$, $\tan 100°$. **6.** π, $\pi/5$, $2\pi/a$, 6π, $8\pi/3$.

7. $a+b-c$. **8.** 0. **9.** $-\sin\theta$, $\cos\theta$, $-\sin\theta$, $\sin\theta$, $\tan\theta$.

10. $(a-b)\sin\theta$. **11.** $(a+b)\cot\theta-(a-b)\tan\theta$. **12.** 1.

Exercises.—IV. Page 21.

1. $2n\pi+\pi/6$, $(2n+1)\pi-\pi/6$. **2.** $n\pi\pm\pi/4$. **3.** $n\pi\pm\alpha$.

5. $(4n+1)\pi/10$. **6.** $(2n+1)\pi/6$. **7.** $4n\pi+2\pi/3$. **8.** $(2n+1)\pi/10$.

9. $(2n+1)\pi/8$. **10.** $(2n+1)\pi/4$, $n\pi/6$.

11. The first equation gives

$\alpha-2y-\pi/2+x=2n\pi$, or $\alpha-2y+\pi/2-x=(2n+1)\pi$.

The second gives $\alpha-3x-\pi/2-\beta+x+3y=2n\pi$,

or $\alpha-3x+\pi/2+\beta-x-3y=(2n+1)\pi$.

12. The equations give

$$x - y = 2n\pi + \pi/6, \text{ or } = (2n+1)\pi - \pi/6, \text{ and } x + y = 2n\pi \pm \pi/6.$$

14. $n\pi \pm \pi/4$. **15.** $(2n+1)\pi/2$. **16.** $n\pi \pm \pi/3$. **17.** $\alpha + n\pi$.

18. $\alpha - \beta - n\pi - \pi/3$. **19.** $2\alpha/11 + (2n+1)\pi/11$.

20. The second equation gives $mx - ny = n\pi + \pi/4$.

21. The second equation gives $2x + 3y = 2n\pi$.

22. The first equation gives

$$2x - 3y = 2n\pi + \pi/2, \text{ or } 2x + 3y = (2n+1)\pi + \pi/2.$$

The second gives $3x - 4y = n\pi + \pi/2 - \alpha$.

23. From the first equation, $4x - 3y = 2n\pi \pm \pi/4$.

From the second, $x + y = n\pi - \pi/4$.

Exercises.—V. Page 27.

1. $\pm 4/5$. **2.** $\pm 15/17$. **3.** $\pm 5/12$. **4.** $\pm 5/4$.

6. (1°) $\sqrt{2mn}/(m+n)$, (2°) $2mn/(m^2+n^2)$, (3°) $\sqrt{m/n}$, (4°) $\sqrt{(n^2-m^2)}/m$.

31. $\sin\theta = 1/2$. **32.** $\tan\theta = 3$, or $1/3$. **33.** $\sin\theta = 0$, or $\cos\theta = 1/2$.

34. $\sec^2\theta = 2$ or 3. **35.** $\sin\theta = 1/3$. **36.** $\sin\theta = 1/2$.

37. $n\pi$, or $2n\pi \pm \pi/3$. **38.** $n\pi \pm \pi/6$.

39. $2\theta = n\pi \pm \alpha$, where $\sin^2\alpha = 2/3$.

40. $2n\pi + \pi/3 \pm \pi/4$. **41.** $n\pi \pm \pi/6$. **42.** $2n\pi + \pi/6$, or $(2n+1)\pi - \pi/6$. **43.** $2n\pi + \pi/3$, or $(2n+1)\pi - \pi/3$.

44. $2n\pi \pm 5\pi/12$.

53. $\sin\theta = (3 \pm \sqrt{9 - 8a})/4$; a must lie between $+1$ and -5.

54. $\tan\theta = \tan\alpha \pm \sec\alpha\sqrt{\cos 2\alpha}$; α must lie between $n\pi + \pi/4$ and $n\pi - \pi/4$.

55. $\tan\theta = 3/4 \pm \sqrt{9/16 - a}$; a cannot be $> 3/4$. **56.** $\tan\theta = 1$, or $1/2$.

57. $\tan x = \cot\alpha \pm \sqrt{\cot^2\alpha - \operatorname{cosec}\alpha}$; α must be $> \sin^{-1}(2\sin 18°)$.

58. $x = (2n+1)\pi \pm \pi/5$. **59.** $\sin x = 2a - 1$, or $1/2a$.

60. $\sin x = (3m \pm \sqrt{9m^2 - 4m})/2$. In order that $\sin x$ may lie between $+1$ and -1, m must be $+$ and > 2, and the radical must be $-$.

61. $\sin^2\theta = (a + b + c \pm \sqrt{(a+b+c)^2 - 4ac})/2a$. If a, c have contrary signs, the radical cannot be taken with the negative sign.

62. $\sin^2 x = (a^2 - b^2)/(1 - b^2)$, $\sin^2 y = (a^2 - b^2)/(a^2 - a^2 b^2)$.

63. $2x = \sin^{-1}(a + b) + \sin^{-1}(a - b)$, $2y = \sin^{-1}(a + b) - \sin^{-1}(a - b)$.

64. $\tan x$, $\tan y$ are the roots of the quadratic $bt^2 - abt + a = 0$.

65. $\tan x$, $\tan y$ are respectively the roots of the equations
$at^2 - (aa' - bb' + 1)\, t + a' = 0$, $b't^2 - (bb' - aa' + 1)\, t + b = 0$.

Exercises.—VI. Page 40.

30. $\tan x = (b - \sin\alpha)/(\cos\alpha - a)$. **31.** $\cos(x - y) = a^2 + b^2 - 2$, $\cos(x + y) = b^2 - a^2$. **32.** $\sin(2x + \pi/4) = \{(a + 1)\sin\pi/4\}/(a - 1)$.

33. $\cot x = 1 + \sqrt{2}/a$. **34.** The values of $\tan x$ are the roots of the quadratio $t^2\cos\alpha - t\sin\alpha + m = 0$. **35.** $2\cos 2x = (m + 1)/(m - 1)$.

36. $\tan x = \sqrt{3}/4$. **37.** $\cos(2x + \pi/4) = (1 - m)/(1 + m)\sqrt{2}$.

38. $\cos 2x = 2/m$. **39.** $\cos^2 x = \sin^2\alpha$, or $1 + 3\cos^2\alpha$.

40. $\sin^2 x$ is a root of the quadratic $24z^2 + 5z = 4$.

Exercises.—VII. Page 43.

23. $x = (2n + 1)\pi/2$, or $\sin^{-1}(a/2)$. **24.** $x = n\pi$, or $\cos x = (-1 \pm \sqrt{17})/4$.

25. $x = n\pi/3$, or $(2n + 1)\pi/2$, or $n\pi$. **26.** $x = n\pi$, or $\sin^{-1}(1/a)$.

27. $\sin x$ is given by the quadratic $2\sin^2 x + 4\sin x = 1$; only one of the roots is admissible, as the other is greater than unity.

28. $x = n\pi - \pi/4$, or $= \frac{1}{2}\sin^{-1}(2\sqrt{2} - 1)$.

29. $\sin x$ is a root of $2\sin^2 x - 3\sin x = 1 - a$.

30. $x = n\pi + \pi/4$, $n\pi - \pi/12$, or $(2n - 1)\pi/2 + \pi/12$.

31. $x = (2n + 1)\pi$, or $\cos^{-1}5/8$. **32.** $\tan x$ is a root of $t^4 + 2t^3 = 3t^2 + 4$.

33. $2x = \tan^{-1}(4/3)$. **34.** $x = n\pi - \pi/4$, or $\pm\sin^{-1}(4/5)$.

35. $\sin 2x = \sqrt{3} - 1$. **36.** $x = 2n\pi$. **37.** $\tan x$ is a root of $t^4 - 9t^2 + 2 = 0$.

38. $x = n\pi \pm \pi/4$, or $\frac{1}{2}\sin^{-1}(1/a)$.

39. $2x = \sin^{-1}(a^2 - 1)$. **40.** $x = n\pi \pm \pi/6$.

EXERCISES.—VIII. Page 53.

1. $\pm\sqrt{3}/2$, $\pm\frac{1}{2}$. **2.** Between $(2n+1)\pi+\pi/4$ and $(2n+1)\pi+3\pi/4$.

3. $\tan\theta = 3/4$, or $-4/3$; $\sin\theta=\pm 3/5$, $\pm 4/5$; $\cos\theta = \pm 4/5$, $\pm 3/5$.

14. $\theta = n\pi - \pi/4$, or $\theta = n\pi$. **15.** $\theta = 3n\pi$, or $\pm\pi/4$.

16. $\theta = 2\cos^{-1} 4a/(1+\sqrt{16a^2+1})$. **17.** $\theta = 2n\pi$, or $\sin^{-1} 1/2a$.

18. $\theta = 2n\pi - \pi/2$, or $\sin^{-1}\{(1-a^2)/a^2\}$. **19.** $\theta = 2n\pi - \pi/2$, or $\sin^{-1}(1-a)$.

20. $2\sin^2\frac{1}{2}\theta = 1 - 1/(1+\tan^2\theta)^{\frac{1}{2}}$, $2\cos^2\frac{1}{2}\theta = 1 + 1/(1+\tan^2\theta)^{\frac{1}{2}}$.

21. $\sin\theta/4$ and $\cos\theta/4$ are roots of $8z^4 - 8z^2 + 1 - \cos\theta = 0$.

22. Put $2\theta/3 = \alpha$, then $2\sin^2\theta = 1 + 3\cos\alpha - 4\cos^3\alpha$,

$$2\cos^2\theta = 1 - 3\cos\alpha + 4\cos^3\alpha.$$

37. $2\theta = 2n\pi \pm \pi/3$. **38.** $\theta = (2n+1)\pi/2$, $2n\pi \pm \pi/3$, $2n\pi \pm 2\pi/3$.

39. $\theta = (2n+1)\pi$, $(4\lambda - 1)\pi/(n-1)$, or $(4\lambda+1)\pi/(n+1)$.

40. $\theta = 2n\pi + \pi/3$. **41.** $\theta = 2n\pi/5$, $(2n+1)\pi/2$, $\cos^{-1}(1\pm\sqrt{17})/8$.

42. $\theta = n\pi$.

EXERCISES.—IX. Page 56.

9. $x = 53° + n\pi$, or $37° + n\pi$. **10.** $x = n\pi \pm \pi/3$.

EXERCISES.—X. Page 61.

12. $4\cos\alpha/2 . \cos\alpha . \sin 5\alpha/2$. **13.** $\sin 3\alpha . \sin 5\alpha/2 . \operatorname{cosec}\alpha/2$.

14. $8\cos(\alpha+\beta+\gamma)/4 . \cos(\beta+\gamma-\alpha)/4 . \cos(\gamma+\alpha-\beta)/4 . \cos(\alpha+\beta+\gamma)/4$
$- 8\sin(\alpha+\beta+\gamma)/4 . \sin(\beta+\gamma-\alpha)/4 . \sin(\gamma+\alpha-\beta)/4 . \sin(\alpha+\beta-\gamma)4$.

15. In Ex. 14, change α, β, γ into $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$.

18. $3x = 2\sin^{-1}(a/2)$. **19.** $x = n\pi$, or $= -2/3 . \sin^{-1}(1/2a)$.

20. $2\sin x$ is a root of $z^3 - 4z + 1 = 0$.

21. $\sin 2x$ is a root of $z^2 + (1-2a)z + a^2 - 1 = 0$.

22. $x = 2n\pi + \pi/6$, $(2n+1)\pi - \pi/6$, $(2n+1)\pi/2$, $2n\pi \pm \pi/3$,
$2n\pi \pm 2\pi/3$.

23. $x = 2n\pi + \pi/6$, $(2n+1)\pi - \pi/6$, $2n\pi + \pi/10$, $(2n+1)\pi - \pi/10$.

Exercises.—XI. Page 66.

13. $x = n$, or $n^2 - n + 1$. **14.** $x^2 = (10 - 4\sqrt{2})/17$. **15.** $x = -1$, or $1/6$.

16. $x = (a + b)/(1 - ab)$. **17.** $x = 0$, or $\pm 1/2$. **18.** $x = ab$.

19. $x = \frac{1}{2}\sin^{-1}(2n - 3)/4$. **20.** $x = \dfrac{1}{\sqrt{2}}$.

Exercises.—XII. Page 70.

1. $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$. **2.** $x^{\frac{2}{3}} y^{\frac{2}{3}} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 1$.

3. $n^2(1 + m^2 \tan^2 a) = \sin^2 a$. **4.** $m^2 n^2 (m^2 - n^2) = 1$.

5. $x^2/a^2 + y^2/b^2 = 1$. **6.** $y = \dfrac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$. **7.** $xy^2 = 4a^3$.

8. If $N = \{\sqrt{k - 12y} - k^2\}$, the result is $k^2x^2 = 2N(N + 3k^2)^2/27$.

9. $a^2x^2 + b^2y^2 = a^2b^2$. **10.** $\cos^2\theta - (\cos\alpha \sin^2\beta + \cos^2\beta)^2 = 0$.

11. $\cos(\alpha - \beta) = (ac + bd)/ad + bc)$. **12.** $\cos^2\theta - \cos^2\alpha \sec^2\beta = 0$.

13. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}$. **14.** $(b \pm \sqrt{b^2 - ac})^2/a^2 + (b' \pm \sqrt{b'^2 - a'c'})^2/a'^2 = 1$.

15. $(a^2 - 1) - (a^2 + 1)\sin\theta = 0$. **16.** $\cos(\theta - \psi) = 1$, or $\cos 2\alpha$.

17. $m^2 + m\cos\alpha = 2$. **18.** $\alpha + \beta + \gamma = \pi$.

19. $x^2 + 4\cot^2\beta = 4\cot\alpha\cot\beta + 1$. **20.** $b^2 = a^2 - 2ac\cos 2\theta + c^2$.

21. $\cos\theta + \cos(n\pi + 2\alpha) = 2c$. **22.** $x^{\frac{4}{5}} + y/\sqrt{(x^{\frac{2}{5}} - 1)} = 2$.

23. $4m^2 = (\sin^{\frac{2}{3}}\alpha + \cos^{\frac{2}{3}}\alpha)^2$. **24.** $2ab(3a^2 + 5b^2) = (a^2 + b^2)^2$.

25. $(bc' - b'c)^2 + (ca' - c'a)^2 = (ab' - a'b)^2$. **26.** $\tan 2\theta = \tan(2\alpha + 2\beta)$.

27. $\cos\alpha(1 + \cos\beta\cos\gamma) = \cos\beta + \cos\gamma$.

28. If we put $\cos\theta\cos\phi = \alpha$, and $\sin\theta\sin\phi = \beta$, we find, from equations (1) and (3),

$$(a + b)\alpha^3 + b\alpha^2 + x^2(a - b)\alpha - bx^2 = 0;$$

and from (2) and (3),

$$(a + b)\beta^3 - (a + c)\beta^2 + y(a - b)\beta + (c - a)y^2 = 0.$$

Hence the result will be got by eliminating α and β from these equations, and $a\alpha + b\beta + c = 0$.

29. $(a^2x^2 + b^2y^2)^3 = c^4(a^2x^2 - b^2y^2)^2$.

30. $\cos\beta(1 + \cos\alpha\cos\alpha') = \cos\alpha + \cos\alpha'$.

31. From equations (1) and (2),

$$\cos^4\theta = q^2/(pq - p^2), \quad \cos^4\phi = p/(q - p).$$

32. $(x - \cos\alpha)^2 + (y - \sin\alpha^2) = 3$. 33. $b^2(x^2 + y^2) = a^2(b^2 + y^2)$.

34. $\cot\alpha = 1/a - 1/b$. 35. $(a - b)(a - b')b^2 = a^2(a' - b)(a' - b')$.

36. $x^2/a^2 + y^2/b^2 = \sec^2\alpha$. 37. $\tan^2\alpha = \tan^2\beta + \tan^2\gamma$.

38. $z^2\{(a - y) + (b - c)\} = (x - c)(y - c)\{(a - y) - (b - c)\}$.

39. $(c - b)x^2 + 2hxy + (c - a)y^2 + 2gx + 2fy + (a + b) = 0$.

Exercises.—XIII. Page 74.

The following rules of transformation were omitted in the text. They supply immediate proofs of several results, and suggest new ones.

In any identity containing the angles, but not the sides of a triangle, the angles may be replaced—

1°. *By* $\pi/2 - A/2, \quad \pi/2 - B/2, \quad \pi/2 - C2.$

2°. *By* $2\pi/3 - A, \quad 2\pi/3 - B, \quad 2\pi/3 - C.$

3°. *By* $\pi/2 - A, \quad \pi/2 - B, \quad \pi - C.$

Thus, from $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$,

we get the three following:—

1°. $\cos A/2 + \cos B/2 + \cos C/2$

$$= 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\cos\left(\frac{\pi - C}{4}\right).$$

2°. $\sin(2\pi/3 - A) + \sin(2\pi/3 - B) + \sin(2\pi/3 - C)$

$$= 4\cos(\pi/3 - A/2)\cos(\pi/3 - B/2)\cos(\pi/3 - C/2).$$

3°. $\cos A + \cos B + \sin C = 4\cos\frac{\pi - 2A}{4}\cos\frac{\pi - 2B}{4}\sin C/2.$

The reason of the rule is obvious; for in each case the sum of the substituted angles is two right angles. The converse of 1° of the foregoing will give another substitution, viz., *for* A, B, C, *we may put*

$$\pi - 2A, \quad \pi - 2B, \quad \pi - 2C.$$

Thus we get

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

Exercises.—XIV. Page 87.

5. $e/2$. **6.** $1/2e$. **7.** $(e - xe^x)/(1 - x)$.

Exercises.—XV. Page 91.

16. 28.

Exercises.—XVI. Page 100.

1.		(1). 2·1072100.	(2). ·7092700.	(3). $\bar{3}$·5051500.
2.		(1). 1·9084852.	(2). 1·3398491.	(3). $\bar{1}$·3856065.
3.		(1). 3·3803920.	(2). ·5352940.	(3). $\bar{1}$·2254900.
7.	1°.	(1). $\bar{3}$·3167252.	(2). ·6354839.	(3). 1·9912260.
		(4). ·8363240.	(5). 1·2375439.	(6). $\bar{3}$·5263393.
	2°.	(1). $\bar{1}$·3116246.	(2). $\bar{1}$·9590810.	(3). $\bar{1}$·7598681.
		(4). ·4322143.	(5). ·2945469.	(6). ·1153521.

8. 31 figures. **16.** $(e^{\sqrt{2}} + e^{-\sqrt{2}} - 4)/8$. **17.** $(e - 1)$.

18. $x = \{\log(2n\pi + m\alpha) - \log\alpha\}/\log 2 - 1$.

Exercises.—XIX. Page 123.

1. $x = 29° \, 21' \, 17''{\cdot}5$. **2.** $x = -(34° \, 20' \, 15'')$. **3.** $2\,L\tan 3x = 10 + L\cos 55° \, 26' \, 34'' + 2L\cos 49° \, 18' \, 36'' - 2L\cos 27° \, 43' \, 17''$.

4. $\tan x = 3{\cdot}5$, or $-{\cdot}\dot{7}1428\dot{5}$. **5.** $\sin x = 3/4$, or $-2/3$.

Exercises.—XX. Page 126.

4. If $\sin^2\phi = 4ab/c^2$, $\tan x = c\cos^2\frac{1}{2}\phi/2a$, or $c\sin^2\frac{1}{2}\phi/2a$.

5. $L\sin 2x = \frac{1}{2}\{20 + \log 2 - \log 3\} = 9{\cdot}9119543$. Hence $x = 27° \, 22' \, 4''$.

6. If $\cos^2\phi = 1/13$, $2\sin x = \sin^2\frac{1}{2}\phi/\cos\phi$, or $-\cos^2\frac{1}{2}\phi/\cos\phi$.

7. If $\tan\phi = 8$, $\sin x = -\cos^2\frac{1}{2}\phi/4\cos\phi$, or $\sin^2\frac{1}{2}\phi/4\cos\phi$.

8. If $\sec^2\phi = 33$, $\cos\frac{1}{2}x = \cos^2\frac{1}{2}\phi/4\cos\phi$, or $-\sin^2\frac{1}{2}\phi/4\cos\phi$.

9. $\sin 2x = -33/49$, or $\sin(45° + x) = 2\sqrt{2}/7$.

10. Put $\rho^2 = 27^2 + 50^2$, $\tan\phi = 27/50$; then x is found from

$$\rho \sin(x + \phi) = 36{\cdot}97.$$

11. $2\cos(a/2 - \pi/4 + y) = b/\sin(a/2 + \pi/4)$.

12. $\sin(a + 2y) = 2b - \sin a$.

13. $2\sin(a + 2y) = b\cos(a + 2y) + b\cos a$, which is of the form (292).

14. $b\cos^2\frac{1}{2}(a - 2x) - 2\cos\frac{1}{2}a\cos\frac{1}{2}(a - 2x) = b\sin^2\frac{1}{2}a$, which is of the form $x^2 - px = q$.

15. $b\cos^2\frac{1}{2}(a + 2y) - 2\sin\frac{1}{2}a\cos\frac{1}{2}(a + 2y) = b\cos^2\frac{1}{2}a$.

16. $(\sin 26° 53' 31''{\cdot}5)\cos\frac{1}{2}(x - y) = 3/7$. This gives $x - y$.

22. $\tan\theta = (-\sin\alpha \pm \sin\beta)/\cos\alpha\cos\beta$. **25.** $\tan^2 x = 1$, or $(a - 1)/(a + 1)$.

26. $\cos 2\theta = \pm\sin 2\alpha$.

27. $2\theta = \sin^{-1}3/10 + \sin^{-1}1/10$, $2\phi = \sin^{-1}3/10 - \sin^{-1}1/10$,

or $2\theta = \sin^{-1}1/30 + \sin^{-1}1/10$, $2\phi = \sin^{-1}1/30 - \sin^{-1}1/10$.

28. $x - y = n\pi \pm \pi/4$, $x + y = (2m + 1)\pi/2$.

Exercises.—XXVIII. Page 153.

1. 50 feet. **2.** 59° 3′ 10″. **3.** 215·6 feet. **4.** 7964 miles.

Exercises.—XXIX. Page 156.

1. $b = 551{\cdot}928$, $A = 30° 24' 51''{\cdot}7$, $B = 59° 35' 8''{\cdot}3$.

2. $b = 1{\cdot}4423$ miles, $A = 15° 57' 43''$, $B = 74° 2' 17''$.

3. $c = 154{\cdot}37$ feet, $A = 38° 10' 43''$, $B = 51° 49' 17''$.

4. $c = 5{\cdot}602$ metres, $A = 40° 39' 24''$, $B = 49° 20' 36''$.

5. $B = 55° 0' 0''$, $b = 6{\cdot}3124$ metres, $c = 7{\cdot}7060$ metres.

6. $A = 20° 30' 0''$, $b = 46{\cdot}378$ yards, $c = 49{\cdot}5135$ yards.

7. $B = 35° 52' 0''$, $a = {\cdot}81038$ miles, $b = {\cdot}58590$ miles.

Exercises.—XXX. Page 160.

1. $A = 27° 47' 45''$, $B = 32° 12' 15''$, $c = 13$.

2. $A = 59° 37' 18''$, $B = 21° 28' 42''$, $c = 590{\cdot}92$.

3. $A = 53° 7' 48''{\cdot}38$. $B = 59° 29' 23''{\cdot}18$, $C = 67° 22' 48''{\cdot}44$, $r = 4$.

4. $B = 90° 0' 0''$, $C = 72° 0' 0''$, $c = 3\sqrt{7 + 2\sqrt{5}}$.

5. $B = 45° 0' 0''$, $C = 120° 0' 0''$, $c = 5\sqrt{6 + 3\sqrt{3}}$.

8. $a = 1000$ yards, $b = 1732{\cdot}05$ yards, $c = 2000$ yards.

9. $A = 119° 26' 51''$, $B = 5° 33' 9''$, **13.** $A = 59° 37' 42''{\cdot}18$.

14. $A = 69° 10' 10''$, $B = 46° 37' 50''$, **16.** $26° 33' 54''$.

17. 9·6733937. **18.** $A = 117° 19' 11''$, $B = 2° 40' 49''$.

19. If θ be the angle of intersection of the common tangents, ϕ of the circles, δ the distance, the intersection of the common tangents, and the intersection of the circles, then

$$\sin \tfrac{1}{2}\theta = (r - r')/d, \quad \cos\phi = (r^2 + r'^2 - d^2)/2rr',$$

$$\delta = 2\sqrt{(s-r)(s-r')\{(r-r')^2 . s . s - d + (r+r')^2(s-r)(s-r')\}}/d(r-r'),$$

where $$2s = d + r + r'.$$

The common tangent is seen from one of the points of intersection of the circles under an angle equal to half that under which the distance between the centres is seen, and from the other point under an angle equal to the supplement of the half.

20. Radius $= \tfrac{1}{2}a \sin\alpha$.

Exercises.—XXXII. Page 173.

1. 180 feet. **15.** Height $= \sqrt{2}t/\sqrt{3}(1 - t)$, where $t = \tan\pi/16$.

16. 4602·5 metres. **18.** Sun's altitude $= \cot^{-1}\sqrt{(b^2/h^2 + b'^2/h'^2)}$.

The inclination of the first wall to the meridian is $\cot^{-1} hb'/h'b$.

19. $AB = 3(3 - \sqrt{3})$.

20. Least distance $= (bu - av)\sin\alpha/\sqrt{u^2 + v^2 - 2uv\cos\alpha}$.

Exercises.—XXXIV. Page 192.

5. Area $= R^2 \sin(p+q)\{\cos(p-q)+\cos(\alpha+\beta)\}$, where

$$\sin p = d\sin\alpha/R, \quad \sin q = d\sin\beta/R.$$

6. If θ be the angle of intersection, and R the circumradius,

$$\theta = \sin^{-1}d/2R - \sin^{-1}b/2R.$$

Exercises.—XXXV. Page 207.

4. $(1+ne^{i\theta})/\sqrt{1+n^2}$, and $(1+ne^{-i\theta})/\sqrt{1+n^2}$.

Exercises.—XXXVI. Page 220.

1. $\pm(\cos 3\alpha + i\sin 3\alpha)$.

2. $\cos\pi/6 + i\sin\pi/6$, $\cos\pi/2 - i\sin\pi/2$, $\cos 5\pi/6 + i\sin 5\pi/6$.

3. If $\tan\alpha = 3/4$, the three roots are $5^{\frac{1}{3}}(\cos\alpha/3 + i\sin\alpha/3)$,

$5^{\frac{1}{3}}(\cos(\alpha+2\pi)/3 + i\sin(\alpha+2\pi)/3)$, $5^{\frac{1}{3}}(\cos(\alpha+4\pi)/3 + i\sin(\alpha+4\pi)/3)$.

Exercises.—XXXVIII. Page 238.

1. $(\tan n\alpha - \tan\alpha)/\sin\alpha$.

2. $(\tan n\alpha - \tan\alpha)/2\sin\alpha$.

3. $(\cot\alpha - \cot(n+1)\alpha)/\cos\alpha$, if n be even;

$(\cot\alpha + \tan(n+1)\alpha)/\cos\alpha$, if n be odd.

4. $n/2 - \cos(2\alpha + \overline{n-1}\,\beta)\sin n\beta/2\sin\beta$.

5. $3\sin(\alpha + \overline{n-1}\,\beta/2)\sin\frac{1}{2}n\beta/4\sin\beta/2$

$$-\sin(3\alpha + 3\overline{(n-1)}\,\beta/2)(\sin 3n\beta/2)/(4\sin 3\beta/2).$$

6. $(n\cos\alpha\sin\alpha - \cos(n+2)\alpha\sin n\alpha)/2\sin\alpha$.

7. $(\cot\alpha - \cot(n+1)\alpha)/\sin\alpha$.

8. $(\tan\alpha - \tan(n+1)\alpha)/\cos\alpha$, if n be even;

$(\tan\alpha + \cot(n+1)\alpha)/\sin\alpha$, if n be odd.

9. $\sin n\alpha \,.\, \sin(n+2)\alpha/2\sin\alpha$, if n be even;

$\sin n\alpha \,.\, \sin(n+2)\alpha/2\sin\alpha - \sin\alpha/2$, if n be odd.

10. $3n/8 + \cos(2\alpha + \overline{n-1}\beta)\sin n\beta/(2\sin\beta)$

$$+ \cos(4\alpha + \overline{2n-2}\beta)\sin 2n\beta/(8\sin 2\beta).$$

11. $\sin\frac{1}{2}(n\alpha)\cos\frac{1}{2}(n+1)\alpha/(2\sin\frac{1}{2}\alpha)$

$$-\sin\frac{3n\alpha}{2}\cos\frac{3}{2}(n+1)\alpha/(2\sin\frac{3}{2}\alpha).$$

12. $\sin n\alpha\cos(n+1)\alpha/\sin\alpha$.

13. This series may be replaced by

$$\sin(\pi-\alpha) + \sin 2(\pi-\alpha) + \sin 3(\pi-\alpha), \text{ \&c.}$$

14. $\cos(2\theta + n\alpha)\sin n\alpha/(2\sin\alpha) + n\cos\alpha/2$.

15. $\sin n\theta\sin(n+1)\theta/(\sin\theta)$.

16. $\sin n\alpha\{\sec\alpha \,.\, \sec(n+1)\alpha + \sec 2\alpha\sec(n+2)\alpha\}/\sin 2\alpha$.

17. $\sin(n\alpha + \frac{1}{2}(n+1)\beta)\sin n(\alpha + \frac{1}{2}\beta)/(2\sin(\alpha + \frac{1}{2}\beta))$

$$-\sin(n\alpha - \tfrac{1}{2}(n+1)\beta)\sin n(\alpha - \tfrac{1}{2}\beta)/(2\sin(\alpha - \tfrac{1}{2}\beta)).$$

18. $n\left\{\sin(\alpha + \overline{n-1}\beta) + \sin(\alpha + \frac{2n-1}{2}\beta)\cos\frac{\beta}{2}\right\}/4\sin^2\frac{\beta}{2}$

$$-n(2n-1)\cos(\alpha + \frac{2n-1}{2}\beta)/4\sin\frac{\beta}{2}$$

$$-\sin(\alpha + \frac{n-2}{2}\beta)\sin\frac{n\beta}{2}\cos\frac{\beta}{2}/2\sin^3\frac{\beta}{2}.$$

19. $n(n+1)\cos(\alpha + \frac{2n-1}{2}\beta)/2\sin\frac{\beta}{2} + \sin\frac{n\beta}{2}\cos(\alpha + \frac{n-2}{2}\beta)/4\sin^2\frac{\beta}{2}$

$$-n\{(2n+1)\cos\frac{\beta}{2}\sin(\alpha + \frac{2n-1}{2}\beta) + n\sin(\alpha + (n-1)\beta)\}/4\sin^2\frac{\beta}{2}$$

$$-n\{\cos\frac{\beta}{2}\cos(\alpha + \overline{n-1}\beta) + \cos^2\frac{\beta}{2}\cos(\alpha + \frac{2n-1}{2}\beta)\}/4\sin^3\frac{\beta}{2}$$

$$+3\cos^2\frac{\beta}{2}\sin\frac{n\beta}{2}\cos(\alpha + \frac{n-2}{2}\beta)/4\sin^4\frac{\beta}{2}.$$

20. $\operatorname{cosec}^2\alpha - (\operatorname{cosec}^2\alpha/2^n)/2^n$. Sum to inf. is $\operatorname{cosec}^2\alpha - \alpha^{-2}$.

21. $2^n\operatorname{cosec}^2 2^n\alpha - \operatorname{cosec}^2\alpha$.

22. $\{2^n\sin\alpha/2^{n-1} - \sin 2\alpha\}/4$.

23. $(\cos\alpha/2^{n-1})/(2^{n-1}\sin\alpha/2^{n-1})^3 - 8\cos 2\alpha/\sin^3 2\alpha$.

24. $4^{n-1}\sin^2\alpha/2^{n-1} - (\sin^2 2\alpha)/4$.

25. Make use of the formula $16\sin^5\alpha = \sin 5\alpha - 5\sin\alpha + 20\sin^3\alpha$.

26. $\{n\cos(2n+1)\alpha/2 \,.\, \sin\alpha/2 - \frac{1}{2}\sin n\alpha\}/(2\sin^2\alpha/2)$.

27. $1 + 2^n\cos^n\frac{1}{2}\alpha\sin\frac{1}{2}n\alpha$. **28.** Differentiate the result in Ex. 26.

29. $x\{\sin\alpha + x\sin(\alpha-\beta) + (-1)^{n+1}(x^n\sin(\alpha+n\beta)$

$$+ x^{n+1}\sin(\alpha+\overline{n-1}\,\beta)\}/(1+2x\cos\beta+x^2).$$

30. $n\,.\,\overline{n+1}/4 + \{n\sin\alpha(\sin\overline{2n+1}\,\alpha) - \sin^2 n\alpha\}/4\sin^2\alpha$. **31.** $\tan^{-1}nx$.

32. $(\sec(2n+1)\theta/2 - \sec\theta/2)/(4\sin\theta/2)$. **33.** $e^{\cos^2\alpha}\cos(\alpha+\sin\alpha\cos\alpha)$.

34. $e^{-\cos\alpha}\sin(\sin\alpha)$. **35.** $\cos(\cos\theta)(e^{\sin\theta}+e^{-\sin\theta})/2$.

36. $e^{x\cos\alpha}\sin(x\sin\alpha)$. **37.** $e^{-\cos\alpha\cos\beta}\cos(\sin\alpha\cos\beta)$.

38. $e^{\cos^2\alpha}\sin(\sin\alpha\cos\alpha)$. **39.** $e^{\sin\alpha\cos\alpha}\cos(\alpha+\sin^2\alpha)$.

40. $\cot^{-1}(\cot\theta + \operatorname{cosec}\theta)$. **41.** $\log(\frac{1}{2}\sec\frac{1}{2}\theta)$.

42. $\log 1/\sqrt{(1-2\cos\alpha\cos\beta+\cos^2\alpha)}$. **43.** $\log\sqrt{(1+2\sin\alpha\cos\beta+\sin^2\alpha)}$.

50. $e^{x\cos\alpha}\sin(\alpha+x\sin\alpha)$. **51.** $e^{-x\cos\alpha}\cos(x\sin\alpha)$.

52. $\tan^{-1}\{e^x\sin x/(1+e^x\cos x)\}$. **53.** $1+\frac{1}{2}(e^{x\cos\alpha}-e^{-x\cos\alpha})\sin(x\sin\alpha)$.

54. Substitute x for h in the example, § 185.

55. $(1-3x\cos\theta)/(1-6x\cos\theta+9x^2)+3(1-2x\cos\theta)/(1-4x\cos\theta+4x^2)$.

56. $14x\sin\alpha/(1-14x\cos\alpha+49x^2) - 6x\sin\alpha/(1-6x\cos\alpha+9x^2)$.

Exercises.—XXXIX. Page 245.

2. $\alpha = m\log\rho - n\gamma$, $\beta = n\log\rho + m\gamma$, where $\rho^2 = a^2+b^2$,

$$\tan\gamma = b/a.$$

3. $(a+bi)^{m+ni} = e^\alpha(\cos\beta + i\sin\beta)$, where $\alpha = m\log\rho - n\gamma$,

$$\beta = n\log\rho + m\gamma.$$

INDEX.

THE END.

From the "EDUCATIONAL TIMES."

"The editor has been very happy in some of the changes he has made. The combination of the general and particular enunciations of each proposition into one is good; and the shortening of the proofs, by omitting the repetitions so common in Euclid, is another improvement. The use of the contra-positive of a proved theorem is introduced with advantage, in place of the *reductio ad absurdum;* while the alternative (or, in some cases, substituted) proofs are numerous, many of them being not only elegant but eminently suggestive. The notes at the end of the book are of great interest, and much of the matter is not easily accessible. The collection of exercises, 'of which there are nearly eight hundred,' is another feature which will commend the book to teachers. To sum up, we think that this work ought to be read by every teacher of Geometry; and we make bold to say that no one can study it without gaining valuable information, and still more valuable suggestions."

From the "JOURNAL OF EDUCATION," *Sept.* 1, 1883.

"In the text of the propositions, the author has adhered, in all but a few instances, to the substance of Euclid's demonstrations, without, however, giving way to a slavish following of his occasional verbiage and redundance. The use of letters in brackets in the enunciations eludes the necessity of giving a second or particular enunciation, and can do no harm. Hints of other proofs are often given in small type at the end of a proposition, and, where necessary, short explanations. The definitions are also carefully annotated. The theory of proportion, Book V., is given in an algebraical form. This book has always appeared to us an exquisitely subtle example of Greek mathematical logic, but the subject can be made infinitely simpler and shorter by a little algebra, and naturally the more difficult method has yielded place to the less. It is not studied in schools, it is not asked for even in the Cambridge Tripos; a few years ago, it still survived in one of the College Examinations at St. John's, but whether the reforming spirit which is dominant there has left it, we do not know. The book contains a very large body of riders and independent geometrical problems. The simpler of these are given in immediate connexion with the propositions to which they naturally attach; the more difficult are given in collections at the end of each book. Some of these are solved in the book, and these include many well-known theorems, properties of orthocentre, of nine-point circle, &c. In every way this edition of Euclid is deserving of commendation. We would also express a hope that everyone who uses this book will afterwards read the same author's 'Sequel to Euclid,' where he will find an excellent account of more modern Geometry."

"MATHESIS," *April*, 1885.

"*A Sequel to Euclid* de M. J. Casey est un de ces livres classiques dont le succès n'est plus à faire. La première édition a paru en 1881, la seconde en 1882, la troisième en 1884, et l'on peut prédire sans crainte de se tromper, qu'elle sera suivie de beaucoup d'autres. C'est un ouvrage analogue aux *Théorèmes et Problèmes de Géométrie* de M. Catalan, et il a les mêmes qualités: il est clair, concis, et renferme beaucoup de matières, sous un petit volume. . . . Le sixième livre de louvrage de Casey est aussi étendu à lui seul que les quatre premiers. Il occupe les pages 67 à 158, c'est-à-dire la moitié du volume. C'est en réalité une *Introduction à la Géométrie Supérieure*. Dans cette partie de l'ouvrage, on rencontre des demonstrations d'une rare élégance dues à M. Casey lui-même."

"MATHESIS," *December*, 1886.

"La troisième édition, publiee en 1884, et dont il a été rendu compte dans ce journal (t. v. 1885, p. 76–78) fait aujourd'hui place à une quatrième, dont la première partie, renfermant 164 pages, 152 figures et 293 exercices, est la reproduction de six chapitres de la précédente édition, car l'auteur n'a pas eu le temps d'en remanier la substance; mais toute la seconde partie, 58 pages, 23 figures et 181 exercices, intitulée: Géométrie récente du triangle, est absolument nouvelle et complétement refondue.

"Elle se divise en six sections consacrées aux études suivantes: 1°. Théorie des points isogonaux et isotomiques, des lignes antiparallèles et des symédianes; 2°. théorie des figures directement semblables; 3°. cercles de Lemoine et de Tucker; 4°. théorie générale d'un système de trois figures semblables; 5°. applications particulières de la théorie des figures directement semblables (au cercle de Brocard et au cercle des neuf points); 6°. théorie des polygones harmoniques.

"Chacune de ces subdivisions, notamment la dernière, est suivei d'exercices intéressants, théorèmes à démontrer ou problèmes à résoudre, parmi lesquels nous devons signaler à l'attention des géomètres les élégantes généralisations dues à MM. Casey, G. Tarry et J. Neuberg dans l'étude des polygones harmoniques.

"Nos lecteurs sont déjà familiarisés avec ces nouvelles théories, et ils connaissent aussi l'éminent géomètre qui, depuis trente ans, fait autorité dans l'enseignement des universités de la Grande-Bretagne. L'ouvrage est à la hauteur de la réputation du maître, et à tous les degrés d'avancement de leurs études, les élèves y trouveront un guide précieux et instructit. Mais cet ouvrage n'est pas seulement utile aux étudiants auxquels il s'addresse; les professeurs y rencontreront des démonstrations nouvelles, une abondante variété de problèmes (près de cinq cents (réellement attrayants et judicieusement choisis, enfin une exposition aussi claire que possible des derniers, progrès de la géométrie du triangle, présentés avec toutes les simplifications désirables.

"Les souhaits que nous avions exprimés au sujet de la troisiéme édition se sont trouvés bientôt réalisés; nous ne pouvons mieux faire que de les reproduire aujourd'hui, persuadé que le nouvel ouvrage sera, comme il le mérité, chaleureusement accueilli et justement apprécié par le public mathématique."

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"This treatise exhibits in a marked degree the qualities which distinguish the author's other works. It is at once compact and comprehensive. . . . Dr. Casey's treatise, indeed, may well accomplish for this generation what Dr. Salmon's did for their fathers, namely, to introduce the young mathematician to the latest developments in the highest departments of the Science. . . . Scarcely any important step is there in the work which he has not simplified, giving one, and sometimes several, original methods. . . . The method of projection is treated of in an entirely original manner, not requiring the consideration of space of three dimensions, the projection being performed in one plane, and the equation of the curve obtained by a simple transformation. The properties of the M'Cay, Neuberg, and Brocard Circles are considered at some length: indeed the latter has now a literature of its own, to which Dr. Casey has largely contributed. . . . There are considerably over a thousand examples, graduated from easy applications of the formulæ to exercises of the highest class of difficulty. Many of these are original, and many are historically interesting."

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"This work will give a new impulse to the study of Analytic Geometry, introduce the student to new and more powerful methods, and greatly enlarge his mathematical horizon. It has been known for some time that Dr. Casey was engaged on a 'Conic Sections,' and people expected that notwithstanding the many works in the field, Dr. Casey's would present a good many valuable novelties; but the work has, we venture to think, excelled all anticipation. . . . Dr. Casey makes a large use of determinants. He introduces them in the very first chapter, and employs them to the end. In the first chapter we find a section devoted to complex variables, and Gauss's geometrical representation of them; and from Clebsch he takes the comparison of 'point' and 'line' co-ordinates. He gives a great development to trilinear co-ordinates and tangential equations, discusses the new circles (Tucker's, Brocard's, Neuberg's, M'Cay's), applies Aronhold's notation to the discussion of the general equation of the second degree, and reproduces from his own Papers in the *Transactions* of learned societies many important theorems and problems relating to inscribed and circumscribed figures, orthogonal conics, and the tact-invariant of two-conics. . . . The work is a noble monument to Dr. Casey's genius, and his mastering of all the resources of Modern Analytic Geometry."

From NATURE.

"Dr. Casey, by the publication of this third treatise, has quite fulfilled the expectations we had formed when we stated some months since that he was engaged upon its compilation. It is a worthy companion of those which have preceded it. It possesses many points of novelty, *i.e.* for the English mathematician. He has from the first introduction of certain recent continental discoveries in geometry taken a warm interest in them, and in the purely geometrical treatment of them, has himself given several beautiful proofs, and has added discoveries of his own. We may here note that this work has met with a very warm welcome in France and Belgium. The author himself has added so much in years now long past to several branches of the subject treated of in the volume under notice—the equation of the circle (and of the conic) touching three circles (three conics), and other properties—that he is specially fitted, by his intimate acquaintance with it, and by his long tuitional experience, to write a book on analytical geometry.

From the EDUCATIONAL TIMES, September, 1886.

"In this book the author has added to those propositions usually met with in Treatises on Analytical Geometry many which we have seen in no other books on the subject; notably extending the equations of circles inscribed in and circumscribed about triangles to polygons of any number of sides, and extending to Conics the properties of circles cutting orthogonally. The demonstrations are concise and neat. In many cases the author has substituted original methods of proof advantageously, and in some has also added the old methods. We would specially note his treatment of the General Equation of the Second Degree, which is more satisfactory than many we have seen. Throughout the book there are numerous exercises on the subject matter, and at the end of each section a collection of problems bearing on that part of the subject. These problems have been obtained from Examination Papers and other sources. They have been selected with much care and judgment. The name of the proposer has in many cases been added, and will cause more interest to be taken in the solution."